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Chern forms of singular metrics on vector bundles ☆

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We study singular hermitian metrics on holomorphic vector bundles, following Berndtsson–Păun. Previous work by Raufi has shown that for such metrics, it is in general not possible to define the curvature as a current with measure coefficients. In this paper we show that despite this, under appropriate codimension restrictions on the singular set of the metric, it is still possible to define Chern forms as closed currents of order 0 with locally finite mass, which represent the Chern classes of the vector bundle.

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1. Introduction

Let X be a complex manifold of dimension n, let $E \to X$ be a rank r holomorphic vector bundle over X, and let h denote a hermitian metric on E. The classical differential geometric study of X through (E, h), revolves heavily around the notion of the curvature associated with h. This approach requires the metric to be smooth (i.e. twice differentiable). However, for line bundles Demailly in [4] introduced the notion of *singular* hermitian metrics, and in a series of influential papers he and others showed how these are a fundamental tool for giving complex algebraic geometry an analytic interpretation.

In [2] Berndtsson and Păun introduced the following notion of singular metrics for vector bundles:

Definition 1.1. Let $E \to X$ be a holomorphic vector bundle over a complex manifold X. A *singular hermitian metric* h on E is a measurable map from the base space X to the space of hermitian forms on the fibers. The hermitian forms are allowed to take the value ∞ at some points in the base (i.e. the norm function $\|\xi\|_h$ is a measurable function with values in $[0, \infty]$), but for any fiber E_x the subset $E_0 := \{\xi \in E_x \ ; \ \|\xi\|_{h(x)} < \infty\}$ has to be a linear subspace, and the restriction of the metric to this subspace must be an ordinary hermitian form.

They also defined what it means for these types of metrics to be curved in the sense of Griffiths:

Definition 1.2. Let $E \to X$ be a holomorphic vector bundle over a complex manifold X and let h be a singular hermitian metric. We say that h is *Griffiths negative* if $\|u\|_h^2$ is plurisubharmonic for any (local) holomorphic section u. Furthermore, we say that h is *Griffiths positive* if the dual metric h^* is Griffiths negative.

Remark 1.3. (i) Strictly speaking, [2] define h to be Griffiths negative if $\log \|u\|_h$ is plurisubharmonic for any holomorphic section u. It is, however, not too difficult to show that these two definitions are equivalent (see e.g. [14], section 2).

(ii) Any singular hermitian metric on a vector bundle E induces a dual metric on the dual bundle E^* (see Lemma 3.1 below). This justifies the notion of Griffiths positivity in Definition 1.2 in terms of duality. □

Definition 1.2 is very natural as these conditions are well-known equivalent properties for smooth metrics.

Although Definition 1.1 is very liberal, as it basically puts no restriction on the metrics, it turns out that Definition 1.2 rules out most of the pathological behavior. For example, we have the following proposition ([14], Proposition 1.3 (ii)):