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On the global behavior of solutions of the Maxwell–Klein–Gordon equations

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ABSTRACT

It is known that the Maxwell–Klein–Gordon equations in $\mathbb{R}^{3+1}$ admit global solutions with finite energy data. In this paper, we present a new approach to study the asymptotic behavior of these global solutions. We show the quantitative energy flux decay of the solutions with data merely bounded in some weighted energy space. We also establish an integrated local energy decay and a hierarchy of r-weighted energy decay. The results in particular hold in the presence of large total charge. This is the first result to give a complete and precise description of the global behavior of large nonlinear charged scalar fields.

1. Introduction

In this paper, we study the asymptotic behavior of solutions to the Maxwell–Klein–Gordon equations on $\mathbb{R}^{3+1}$ with large Cauchy data. To define the equations, let $A = A_\mu dx^\mu$ be a 1-form. The covariant derivative associated to this 1-form is

$$D_\mu = \partial_\mu + \sqrt{-1}A_\mu,$$

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which can be viewed as a $U(1)$ connection on the complex line bundle over $\mathbb{R}^{3+1}$ with the standard flat metric $m_{\mu\nu}$. Then the curvature 2-form F is given by

$$F_{\mu\nu} = -\sqrt{-1}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu = (dA)_{\mu\nu}.$$

This is a closed 2-form, that is, F satisfies the Bianchi identity

$$\partial_\gamma F_{\mu\nu} + \partial_\mu F_{\nu\gamma} + \partial_\nu F_{\gamma\mu} = 0. \tag{1}$$

The Maxwell–Klein–Gordon equations (MKG) is a system for the connection field A and the complex scalar field ϕ:

$$\begin{cases} \partial^\nu F_{\mu\nu} = \Im(\phi \cdot \overline{D_\mu \phi}) = J_\mu; \\ D^\mu D_\mu \phi = \Box_A \phi = 0. \end{cases} \tag{MKG}$$

These are Euler–Lagrange equations of the functional

$$L[A, \phi] = \iint\limits_{\mathbb{R}^{3+1}} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi \overline{D^\mu \phi} dx dt.$$

A basic feature of this system is that it is invariant under the following gauge transformation:

$$\phi \mapsto e^{i\chi}\phi; \quad A \mapsto A - d\chi.$$

More precisely, if (A, ϕ) solves (MKG), then $(A - d\chi, e^{i\chi}\phi)$ is also a solution for any potential function χ. Note that $U(1)$ is abelian. The Maxwell field F is invariant under the above gauge transformation and (MKG) is said to be an *abelian gauge theory*. For the more general theory when $U(1)$ is replaced by a general compact Lie group, the corresponding equations are referred to as *Yang–Mills–Higgs equations*.

In this paper, we consider the Cauchy problem to (MKG). The initial data set (E, H, ϕ_0, ϕ_1) consists of the initial electric field E, the magnetic field H, together with initial data (ϕ_0, ϕ_1) for the scalar field. In terms of the solution (F, ϕ), on the initial hypersurface, these are:

$$F_{0i} = E_i, \quad {}^*F_{0i} = H_i, \quad \phi(0, x) = \phi_0, \quad D_t\phi(0, x) = \phi_1,$$

where ${}^*F$ is the Hodge dual of the 2-form F. In local coordinates (t, x),

$$(H_1, H_2, H_3) = (F_{23}, F_{31}, F_{12}).$$

The data set is said to be *admissible* if it satisfies the compatibility condition

$$div(E) = \Im(\phi_0 \cdot \overline{\phi_1}), \quad div(H) = 0,$$