# Busemann's intersection inequality in hyperbolic and spherical spaces 

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## A B S T R A C T

Busemann's intersection inequality asserts that the only maximizers of the integral $\int_{S^{n-1}}\left|K \cap \xi^{\perp}\right|^{n} d \xi$ among all convex bodies of a fixed volume in $\mathbb{R}^{n}$ are centered ellipsoids. We study this question in the hyperbolic and spherical spaces, as well as general measure spaces.
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## 1. Introduction

Let $K$ be a convex body in $\mathbb{R}^{n}$ that contains the origin in its interior. The following is known as Busemann's intersection inequality:

$$
\begin{equation*}
\int_{S^{n-1}}\left|K \cap \xi^{\perp}\right|^{n} d \xi \leq c_{n}|K|^{n-1} \tag{1}
\end{equation*}
$$

with equality if and only if $K$ is a centered ellipsoid; see [1]. Here, $c_{n}=n \kappa_{n-1}^{n} / \kappa_{n}^{n-2}$, where $\kappa_{n}$ is the volume of the unit Euclidean ball $B^{n}$ in $\mathbb{R}^{n}$, and $|A|$ stands for the volume (in the appropriate dimension) of a set $A$.

In fact, the inequality (1) is true for a larger class of sets, in particular, star bodies; see [3, p. 373]. In a slightly different form, (1) can be stated as follows. Centered ellipsoids in $\mathbb{R}^{n}$ are the only maximizers of the quantity

$$
\begin{equation*}
\int_{S^{n-1}}\left|K \cap \xi^{\perp}\right|^{n} d \xi \tag{2}
\end{equation*}
$$

in the class of star bodies of a fixed volume. In this paper we study this question in the hyperbolic space $\mathbb{H}^{n}$ and the sphere $\mathbb{S}^{n}$ (or, more precisely, a hemisphere $\mathbb{S}_{+}^{n}$, as explained in the next section). We show that in $\mathbb{H}^{n}$ centered balls are the unique maximizers of (2) in the class of star bodies of a fixed volume. On the sphere the situation is different. In $\mathbb{S}_{+}^{2}$ centered balls are in fact the unique minimizers (in the class of origin-symmetric star bodies). The maximizers of (2) in the class of origin-symmetric star bodies in $\mathbb{S}_{+}^{2}$ are cones (see Section 2 for the definition). The maximizers of (2) in the class of origin-symmetric convex bodies in $\mathbb{S}_{+}^{2}$ are lunes. It is surprising that in $\mathbb{S}_{+}^{n}$ with $n \geq 3$ centered balls are neither maximizers nor minimizers, even in the class of origin-symmetric convex bodies. We also obtain an optimal lower bound for (2) in the class of star bodies in $\mathbb{S}_{+}^{n}, n \geq 3$, of a given volume and describe the equality cases. Finally, we prove a version of Busemann's intersection inequality (together with the equality cases) for general measures on $\mathbb{R}^{n}$ and $\mathbb{H}^{n}$. More precisely, let $\mu$ be a measure on $\mathbb{R}^{n}$ or $\mathbb{H}^{n}$ with density that is radially symmetric and decreasing. Then the maximum of $\int_{S^{n-1}} \mu\left(K \cap \xi^{\perp}\right)^{n} d \xi$ in the class of star bodies of a fixed measure $\mu$ is given by the geodesic balls centered at the origin.

For the history of Busemann's inequality, its applications, and recent developments the reader is referred to [2], [3], [4], [5], [8].

It is interesting to note that, in the context of the Busemann-Petty problem (see [7] for the statement and history of this problem), the sphere and the Euclidean space are similar in the sense that the positive answer holds in the same dimensions, while the hyperbolic space exhibits a different behavior; see [9]. For Busemann's intersection inequality, the hyperbolic space is similar to the Euclidean space, while the sphere is not. It is also worth mentioning that the answer to the Busemann-Petty problem for arbitrary

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