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Rigorous effective bounds on the Hausdorff dimension of continued fraction Cantor sets: A hundred decimal digits for the dimension of E_2

O. Jenkinson [a,*], M. Pollicott [b]

[a] *School of Mathematical Sciences, Queen Mary University of London, London E1 4NS, UK*
[b] *Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK*

A R T I C L E   I N F O

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A B S T R A C T

We prove that the algorithm of [19] for approximating the Hausdorff dimension of dynamically defined Cantor sets, using periodic points of the underlying dynamical system, can be used to establish completely rigorous high accuracy bounds on the dimension. The effectiveness of these rigorous estimates is illustrated for Cantor sets consisting of continued fraction expansions with restricted digits. For example the Hausdorff dimension of the set E_2 (of those reals whose continued fraction expansion only contains digits 1 and 2) can be rigorously approximated, with an accuracy of over 100 decimal places, using points of period up to 25.

The method for establishing rigorous dimension bounds involves the holomorphic extension of mappings associated to the allowed continued fraction digits, an appropriate disc which is contracted by these mappings, and an associated transfer operator acting on the Hilbert Hardy space of analytic functions on this disc. We introduce methods for rigorously bounding the approximation numbers for the transfer operators, showing that this leads to effective estimates on the

* Corresponding author.
E-mail addresses: o.jenkinson@qmul.ac.uk (O. Jenkinson), M.Pollicott@warwick.ac.uk (M. Pollicott).

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Taylor coefficients of the associated determinant, and hence to explicit bounds on the Hausdorff dimension.

1. Introduction

For a finite subset $A \subset \mathbb{N}$, let E_A denote the set of all $x \in (0,1)$ such that the digits $a_1(x), a_2(x), \ldots$ in the continued fraction expansion

$$x = [a_1(x), a_2(x), a_3(x), \ldots] = \cfrac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \cdots}}}$$

all belong to A. Sets of the form E_A are said to be of *bounded type* (see e.g. [20,23]); in particular they are Cantor sets, and study of their Hausdorff dimension has attracted significant attention.

Of particular interest have been the sets $E_n = E_{\{1,\ldots,n\}}$, with $E_2 = E_{\{1,2\}}$ the most studied of these, serving as a test case for various general methods of approximating Hausdorff dimension. Jarnik [18] showed that $\dim(E_2) > 1/4$, while Good [15] improved this to $0.5306 < \dim(E_2) < 0.5320$, Bumby [6] showed that $0.5312 < \dim(E_2) < 0.5314$, Hensley [16] showed that $0.53128049 < \dim(E_2) < 0.53128051$, while Falk & Nussbaum [11][1] rigorously justified the first 8 decimal digits of $\dim(E_2)$, proving that $0.5312805505981423 \leq \dim(E_2) \leq 0.5312805506343388$. A common element in the methods [6,11,16] is the study of a transfer operator, while for the higher accuracy estimates [11,16] there is some element of computer-assistance involved in the proof.

In [19] we outlined a different approach to approximating the Hausdorff dimension of bounded type sets, again using a transfer operator, but exploiting the real analyticity of the maps defining continued fractions to consider the determinant Δ of the operator, and its approximation in terms of periodic points[2] of an underlying dynamical system. While some highly accurate *empirical* estimates of Hausdorff dimension were given, for example a 25 decimal digit approximation to $\dim(E_2)$, these were not rigorously justified. Moreover, although the algorithm was proved to generate a sequence of approximations s_n to the Hausdorff dimension (depending on points of period up to n), with convergence rate faster than any exponential, the derived error bounds were sufficiently conservative (see Remark 1 below) that it was unclear whether they could be combined with the computed approximations to yield any effective *rigorous* estimate.

[1] This preprint has been split into the two articles [12] and [13], with [12] containing the approximation to $\dim(E_2)$.

[2] The periodic points are precisely those numbers in $(0,1)$ with periodic continued fraction expansion, drawn from digits in A. The reliance on periodic points renders the method *canonical*, inasmuch as it does not involve any arbitrary choice of coordinates or partition of the space.