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Gap-labelling conjecture with nonzero magnetic field

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ABSTRACT

Given a constant magnetic field on Euclidean space $\mathbb{R}^p$ determined by a skew-symmetric $(p \times p)$ matrix Θ, and a $\mathbb{Z}^p$-invariant probability measure μ on the disorder set Σ which is by hypothesis a Cantor set, where the action is assumed to be minimal, the corresponding Integrated Density of States of any self-adjoint operator affiliated to the twisted crossed product algebra $C(\Sigma) \rtimes_\sigma \mathbb{Z}^p$, where σ is the multiplier on $\mathbb{Z}^p$ associated to Θ, takes on values on spectral gaps in the *magnetic gap-labelling group*. The *magnetic frequency group* is defined as an explicit countable subgroup of $\mathbb{R}$ involving Pfaffians of Θ and its sub-matrices. We conjecture that the magnetic gap labelling group is a subgroup of the magnetic frequency group. We give evidence for the validity of our conjecture in 2D, 3D, the Jordan block diagonal case and the periodic case in all dimensions.

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Contents

1. Introduction

The gap-labelling theorem was originally conjectured by Bellissard [5] in the late 1980s. It concerns the labelling of gaps in the spectrum of a Schrödinger operator (in the absence of a magnetic field) by the elements of a subgroup of $\mathbb{R}$ which results from pairing the K_0-group of the noncommutative analog for the Brillouin zone with the tracial state defined by the probability measure on the hull. The problem arises in a mathematical version of solid state physics in the context of aperiodic tilings. Its three proofs, discovered independently by the authors of [15,28,6] all concern the proof of a statement in K-theory. Earlier results include the proof of the gap-labelling conjecture in 1D [7], 2D [8,49] and in 3D [9]. A more detailed account of the history of gap-labelling theorems can be found in Appendix B.

In the presence of a non-zero constant magnetic field in Euclidean space, the gap-labelling conjecture is much trickier to state, even though it was known to be the more interesting problem in spectral theory and in condensed matter physics since the 1980s, cf. [4]. Here, we manage to give, for the first time, a precise formulation of conjectures for the magnetic gap-labelling group in all dimensions which encompass all previously known results. More precisely, in this paper we initiate the study of the gap-labelling group in the case of the magnetic Schrödinger operator on Euclidean space $\mathbb{R}^p$ with disorder set a Cantor set Σ under a non-zero magnetic field $B = \frac{1}{2} dx^t \Theta dx$, where Θ is a $(p \times p)$ skew-symmetric matrix. We believe that proving (or disproving) our conjectures would constitute an important step in the understanding of aperiodic tilings under a constant magnetic field. Given a $\mathbb{Z}^p$-invariant probability measure μ on Σ, the corresponding Integrated Density of States of any self-adjoint operator affiliated to the twisted crossed product algebra $C(\Sigma) \rtimes_\sigma \mathbb{Z}^p$ takes values on spectral gaps in an explicit