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# Building blocks of polarized endomorphisms of normal projective varieties

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ABSTRACT

An endomorphism $f$ of a projective variety $X$ is polarized (resp. quasi-polarized) if $f^*H \sim qH$ (linear equivalence) for some ample (resp. nef and big) Cartier divisor $H$ and integer $q > 1$. First, we use cone analysis to show that a quasi-polarized endomorphism is always polarized, and the polarized property descends via any equivariant dominant rational map. Next, we show that a suitable maximal rationally connected fibration (MRC) can be made $f$-equivariant using a construction of N. Nakayama, that $f$ descends to a polarized endomorphism of the base $Y$ of this MRC and that this $Y$ is a $Q$-abelian variety (quasi-étale quotient of an abelian variety). Finally, we show that we can run the minimal model program (MMP) $f$-equivariantly for mildly singular $X$ and reach either a $Q$-abelian variety or a Fano variety of Picard number one.
As a consequence, the building blocks of polarized endomorphisms are those of $Q$-abelian varieties and those of Fano varieties of Picard number one.
Along the way, we show that $f$ always descends to a polarized endomorphism of the Albanese variety $\mathrm{Alb}(X)$ of $X$, and that the pullback of a power of $f$ acts as a scalar multiplication on the Néron–Severi group of $X$ (modulo torsion) when $X$ is smooth and rationally connected.

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Partial answers about $X$ being of Calabi–Yau type, or Fano type are also given with an extra primitivity assumption on $f$ which seems necessary by an example.


**Contents**



## 1. Introduction

We work over an algebraically closed field $k$ which has characteristic zero, and is uncountable (only used to guarantee the birational invariance of the rational connectedness property). Let $f$ be a surjective endomorphism of a projective variety $X$. We say that $f$ is *polarized* (resp. *quasi-polarized*), if there is an ample (resp. nef and big) Cartier divisor $H$ such that $f^*H \sim qH$ (linear equivalence) for some integer $q > 1$. If $X$ is a point, then the only trivial endomorphism is polarized by convention.

Let $X$ be a projective variety of dimension $n$. We refer to Definition 2.1 for the numerical equivalence ($\equiv$) of $\mathbb{R}$-Cartier divisors and Definition 2.2 for the weak numerical equivalence ($\equiv_w$) of $r$-cycles with real coefficients. Denote by $\mathrm{N}^1(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ for the Néron–Severi group $\mathrm{NS}(X)$. One can also regard $\mathrm{N}^1(X)$ as the quotient vector space of $\mathbb{R}$-Cartier divisors modulo the numerical equivalence; see Definition 2.1. Denote by $\mathrm{N}_r(X)$ the quotient vector space of $r$-cycles modulo the weak numerical equivalence.

Suppose further $X$ is normal. Then the numerical equivalence and the weak numerical equivalence are the same for $\mathbb{R}$-Cartier divisors; in particular, the natural map $\mathrm{N}^1(X) \to \mathrm{N}_{n-1}(X)$ is well defined and an injection (cf. Definition 2.2 and Lemma 2.3). A Weil $\mathbb{R}$-divisor $F$ is said to be *big* if $F = A + E$ for some ample $\mathbb{Q}$-Cartier divisor $A \in \mathrm{N}^1(X)$ and pseudo-effective Weil $\mathbb{R}$-divisor $E$; see Definition 2.4.

A surjective endomorphism $f : X \to X$ of a projective variety $X$ is a finite morphism. In fact, $f$ induces an automorphism $f^* : \mathrm{N}^1(X) \to \mathrm{N}^1(X)$. So an ample divisor is the pull back of some divisor, which, together with the projection formula, imply the finiteness of $f$. Suppose further $f^*H \sim qH$ for some nef and big Cartier divisor $H$ and $q > 0$, then,