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A vanishing theorem for Dirac cohomology of standard modules

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A B S T R A C T

This paper studies the Dirac cohomology of standard modules in the setting of graded Hecke algebras with geometric parameters. We prove that the Dirac cohomology of a standard module vanishes if and only if the module is not twisted-elliptic tempered. The proof makes use of two deep results. One is some structural information from the generalized Springer correspondence obtained by S. Kato and Lusztig. Another one is a computation of the Dirac cohomology of tempered modules by Barbasch–Ciubotaru–Trapa and Ciubotaru.

We apply our result to compute the Dirac cohomology of ladder representations for type A_n. For each of such representations with non-zero Dirac cohomology, we associate to a canonical Weyl group representation. We use the Dirac cohomology to conclude that such representations appear with multiplicity one.

1. Introduction

1.1. Overview

The Dirac operator plays an important role in the representation theory for real reductive groups. It was used for geometric constructions of discrete series by the work of

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Parthasarathy [47] and Atiyah–Schmid [2]. The notion of Dirac cohomology was introduced by Vogan [54] in around 1997 along with a conjecture relating to the infinitesimal characters of Harish-Chandra modules. The conjecture was later proved by Huang–Pandžić [28], and has been extended to many other settings e.g. [29], [35], [6], [16], [12]. The problem of determining the Dirac cohomology of interesting modules has been considered in [30,29,32,24,8] and many others, and also has applications to branching rules [31], and harmonic analysis and endoscopy theory [27].

Graded Hecke algebras were introduced by Lusztig [41,42] for the study of p-adic reductive groups. Motivated by some analogies between p-adic and real reductive groups, Barbasch–Ciubotaru–Trapa [6] define a Dirac operator and develop the Dirac cohomology theory for graded Hecke algebras. The theory has been further developed in [22,23,21] and there are applications on W-character formulas [22,17] and a classification of discrete series [20].

This paper studies the Dirac cohomology of standard modules for graded Hecke algebras. Determining their Dirac cohomology requires understanding on some fine structure. It turns out that part of the information can be obtained from results of generalized Springer correspondence by Kato [34] and Lusztig [40,41,43,44] via the geometric realization of standard modules. More details and applications of our result will be discussed in next two sections.

1.2. Main results

For any complex reductive group H, let H° be the identity component of H.

We first introduce some notations. Let G be a complex connected reductive group with its Lie algebra $\mathfrak{g}$. Let $\mathcal{N}_G$ be the set of nilpotent elements in $\mathfrak{g}$. Let L be a Levi subgroup of G and let $\mathcal{O}$ be a L-orbit in $\mathcal{N}_G$. Let $\mathcal{L}$ be a cuspidal local system of $\mathcal{O}$. The datum $(L, \mathcal{O}, \mathcal{L})$ forms a cuspidal triple. Let P be a parabolic subgroup with the Levi decomposition LU. Let $T = Z_L^\circ$, where Z_L is the center of L. Let $\mathfrak{h}$ be the Lie algebra of T. Let $\mathfrak{h}^\vee$ be the dual space of $\mathfrak{h}$. All these data determine a Weyl group W, a root system R, a set Π of simple roots in R and a parameter function $c : \Pi \to \mathbb{R}$ (see Section 4.1 for the detailed notations). Lusztig [41,43] constructed geometrically a graded Hecke algebra $\mathbb{H}$, which also has an algebraic description $\mathbb{H}(V, W, \Pi, \mathbf{r}, c)$ (see Section 2.1).

Before defining the Dirac cohomology, we introduce more notations. Let $V' = \mathbb{C} \otimes_{\mathbb{Z}} R$. Let $C(V')$ be the Clifford algebra for V' (Section 2.3). Let S be a fixed choice of simple module of $C(V')$. Let $\widetilde{W}$ be the spin double cover of W (Section 2.4). $\widetilde{W}$ then defines a twisted group algebra $\mathbb{C}[\widetilde{W}]$, which is a natural subalgebra of $C(V')$. There is also a diagonal embedding, denoted $\Delta : \mathbb{C}[\widetilde{W}] \to \mathbb{H} \otimes C(V')$, which plays an important role in the Dirac cohomology.

The Dirac element, denoted D, for $\mathbb{H}$ is an element in $\mathbb{H} \otimes C(V')$ which has a remarkable formula for D^2 (2.1). Given an $\mathbb{H}$-module X, define the Dirac cohomology of X (as in [6]):