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Divisor problem in arithmetic progressions modulo a prime power

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ABSTRACT

We obtain an asymptotic formula for the average value of the divisor function over the integers $n \le x$ in an arithmetic progression $n \equiv a \bmod q$, where $q = p^k$ for a prime $p \ge 3$ and a sufficiently large integer k. In particular, we break the classical barrier $q \le x^{2/3-\varepsilon}$ (with an arbitrary $\varepsilon > 0$) for such formulas, and, using some new arguments, generalise and strengthen a recent result of R. Khan (2015), making it uniform in k.

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1. Introduction

1.1. Background

For a positive integer n, let $d(n)$ be the classical divisor function, which is the number of divisors of n. Let a and q be integers with $q \geq 1$ and $\gcd(a,q) = 1$. For $X \geq 2$, define

$$D(X;q,a) := \sum_{\substack{n \leq X \\ n \equiv a \bmod q}} d(n)$$

and also

$$E(X;q,a) := D(X;q,a) - \frac{1}{\varphi(q)} \sum_{\substack{n \leq X \\ \gcd(n,q)=1}} d(n),$$

where φ is the Euler function. In unpublished works, it has been discovered independently by Selberg and Hooley that for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for a sufficiently large X

$$\max_{\gcd(a,q)=1} |E(X;q,a)| \leq \frac{X^{1-\delta}}{q} \tag{1.1}$$

holds uniformly for $q \leq X^{2/3-\varepsilon}$. This follows from the Weil bound for Kloosterman sums, see [16].

When q is large, there are various results on the average bound of $E(X;q,a)$. Fouvry [3, Corollary 5] has studied the average over q and shown that for any $\varepsilon > 0$ there exist some constant $c > 0$ such that for a sufficiently large X and for any $a \in \mathbb{Z}$ with $|a| \leq \exp(c\sqrt{\log X})$ we have

$$\sum_{\substack{X^{2/3+\varepsilon} \leq q \leq X^{1-\varepsilon} \\ \gcd(q,a)=1}} |E(X;q,a)| \leq X \exp(-c\sqrt{\log X}).$$

Banks, Heath-Brown and Shparlinski [1] have considered the average over a and proved that for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for a sufficiently large X

$$\sum_{\substack{1 \leq a \leq q \\ \gcd(a,q)=1}} |E(X;q,a)| \leq X^{1-\delta}$$

holds uniformly for $q \leq X^{1-\varepsilon}$. For other examples, see [2,4,6,7,15].

Irving [8] first has broken through the range given by the Weil bound (see [9, Corollary 11.12]) for some special individual modulus q and proved that, for any $\varpi, \varrho > 0$ satisfying $246\varpi + 18\varrho < 1$, there exists some $\delta > 0$, depending only on ϖ and ϱ such that (1.1) holds uniformly for any X^{ϱ}-smooth, squarefree moduli $q \leq X^{2/3+\varpi}$. Khan [10]