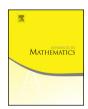


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### Advances in Mathematics





# Divisor problem in arithmetic progressions modulo a prime power



Kui Liu<sup>a</sup>, Igor E. Shparlinski<sup>b</sup>, Tianping Zhang<sup>c,\*</sup>

- <sup>a</sup> School of Mathematics and Statistics, Qingdao University, No. 308, Ningxia Road, Shinan, Qingdao, Shandong, 266071, PR China
- <sup>b</sup> Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052, Australia
- $^{\rm c}$  School of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710119 Shaanxi, PR China

#### ARTICLE INFO

Article history:
Received 28 August 2016
Received in revised form 4 December 2017
Accepted 7 December 2017
Available online xxxx
Communicated by Kartik Prasanna

MSC:

 $11L05 \\ 11N25$ 

11N25 11N37

11T23

Keywords:
Divisor problem
Arithmetic progressions
Kloosterman sums
Prime powers

#### ABSTRACT

We obtain an asymptotic formula for the average value of the divisor function over the integers  $n \leq x$  in an arithmetic progression  $n \equiv a \mod q$ , where  $q = p^k$  for a prime  $p \geq 3$  and a sufficiently large integer k. In particular, we break the classical barrier  $q \leq x^{2/3-\varepsilon}$  (with an arbitrary  $\varepsilon > 0$ ) for such formulas, and, using some new arguments, generalise and strengthen a recent result of R. Khan (2015), making it uniform in k.

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E-mail addresses: liukui@qdu.edu.cn (K. Liu), igor.shparlinski@unsw.edu.au (I.E. Shparlinski), tpzhang@snnu.edu.cn (T.P. Zhang).

<sup>\*</sup> Corresponding author.

#### 1. Introduction

#### 1.1. Background

For a positive integer n, let d(n) be the classical divisor function, which is the number of divisors of n. Let a and q be integers with  $q \ge 1$  and  $\gcd(a, q) = 1$ . For  $X \ge 2$ , define

$$D(X;q,a) := \sum_{\substack{n \le X \\ n \equiv a \bmod q}} d(n)$$

and also

$$E(X;q,a) := D(X;q,a) - \frac{1}{\varphi(q)} \sum_{\substack{n \leq X \\ \gcd(n,q) = 1}} d(n),$$

where  $\varphi$  is the Euler function. In unpublished works, it has been discovered independently by Selberg and Hooley that for any  $\varepsilon > 0$  there exists some  $\delta > 0$  such that for a sufficiently large X

$$\max_{\gcd(a,q)=1} |E(X;q,a)| \le \frac{X^{1-\delta}}{q} \tag{1.1}$$

holds uniformly for  $q \leq X^{2/3-\varepsilon}$ . This follows from the Weil bound for Kloosterman sums, see [16].

When q is large, there are various results on the average bound of E(X; q, a). Fouryy [3, Corollary 5] has studied the average over q and shown that for any  $\varepsilon > 0$  there exist some constant c > 0 such that for a sufficiently large X and for any  $a \in \mathbb{Z}$  with  $|a| \le \exp(c\sqrt{\log X})$  we have

$$\sum_{\substack{X^{2/3+\varepsilon} \le q \le X^{1-\varepsilon} \\ \gcd(a,a)=1}} |E(X;q,a)| \le X \exp(-c\sqrt{\log X}).$$

Banks, Heath-Brown and Shparlinski [1] have considered the average over a and proved that for any  $\varepsilon > 0$  there exists some  $\delta > 0$  such that for a sufficiently large X

$$\sum_{\substack{1 \le a \le q \\ \gcd(a,q)=1}} |E(X;q,a)| \le X^{1-\delta}$$

holds uniformly for  $q \leq X^{1-\varepsilon}$ . For other examples, see [2,4,6,7,15].

Irving [8] first has broken through the range given by the Weil bound (see [9, Corollary 11.12]) for some special individual modulus q and proved that, for any  $\varpi, \varrho > 0$  satisfying  $246\varpi + 18\varrho < 1$ , there exists some  $\delta > 0$ , depending only on  $\varpi$  and  $\varrho$  such that (1.1) holds uniformly for any  $X^\varrho$ -smooth, squarefree moduli  $q \leq X^{2/3+\varpi}$ . Khan [10]

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