Advances in Mathematics 325 (2018) 505–532

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

On a modularity conjecture of Andrews, Dixit, Schultz, and Yee for a variation of Ramanujan's $\omega(q)$ ☆

Kathrin Bringmann [a,*], Chris Jennings-Shaffer [a], Karl Mahlburg [b]

[a] *Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany*
[b] *Department of Mathematics, Louisiana State University, Baton Rouge, LA, USA*

A R T I C L E   I N F O

Article history:
Received 10 January 2017
Received in revised form 28 November 2017
Accepted 28 November 2017
Available online 11 December 2017
Communicated by George Andrews

Keywords:
Integer partitions
Andrews' smallest parts functions
Modular forms
Harmonic Maass forms
Mock theta functions

A B S T R A C T

We analyze the mock modular behavior of $\bar{P}_\omega(q)$, a partition function introduced by Andrews, Dixit, Schultz, and Yee. This function arose in a study of smallest parts functions related to classical third order mock theta functions, one of which is $\omega(q)$. We find that the modular completion of $\bar{P}_\omega(q)$ is not simply a harmonic Maass form, but is instead the derivative of a linear combination of products of various harmonic Maass forms and theta functions. We precisely describe its behavior under modular transformations and find that the image under the Maass lowering operator lies in a relatively simpler space.

☆ The research of the first author is supported by the Alfried Krupp Prize for Young University Teachers of the Krupp foundation and the research leading to these results receives funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013) / ERC Grant agreement No. 335220 – AQSER.

* Corresponding author.
E-mail addresses: kbringma@math.uni-koeln.de (K. Bringmann), jennichr@math.oregonstate.edu (C. Jennings-Shaffer), mahlburg@mat.lsu.edu (K. Mahlburg).

https://doi.org/10.1016/j.aim.2017.12.002
0001-8708/

1. Introduction and statement of results

Andrews introduced the smallest parts partition function $\operatorname{spt}(n)$ in [2], which enumerates the partitions of n weighted by the multiplicity of their smallest parts. In the following years, a large volume of subsequent work has established its importance as a rich source for study, with many interesting examples of combinatorial and algebraic results [12,18], connections to modular forms [11], as well as generalized families of spt-functions [9,13]. A particularly striking feature of the smallest parts function is found in the automorphic properties of its generating function [5]. Indeed, the generating function provides a natural example of a *mock theta function*, in the modern sense; i.e., the holomorphic part of a harmonic Maass form [8,23]. The classical notion of a mock theta function was based on Ramanujan's last letter to Hardy [22], and these functions are now understood in the framework of real-analytic modular forms thanks to Zwegers' seminal Ph.D. thesis [24]. In particular, in Theorem 4 of [2], Andrews showed that

$$\sum_{n\geq 1} \operatorname{spt}(n)q^n = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}\frac{nq^n}{1-q^n} + \frac{1}{(q;q)_\infty}\sum_{n\geq 1}\frac{(-1)^n q^{\frac{n(3n+1)}{2}}(1+q^n)}{(1-q^n)^2}, \tag{1.1}$$

where throughout the paper we use the standard q-factorial notation

$$(a)_n = (a;q)_n := \prod_{j=0}^{n-1}\left(1-aq^j\right)$$

for $n \in \mathbb{N}_0 \cup \{\infty\}$. The key feature of (1.1) is that (up to rational powers of q) it expresses the generating function as a derivative of a linear combination of theta functions and Appell–Lerch sums, which are the main components of Zwegers' work [24] (see Section 2 below).

In a recent paper [4], Andrews, Dixit, and Yee considered the question of constructing smallest parts partition functions directly from mock theta functions, and proved new results arising from the classical mock theta functions $\omega(q)$, $\nu(q)$, and $\phi(q)$. Originally ω is defined as

$$\omega(q) := \sum_{n\geq 0}\frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2};$$

in Theorem 3.1 of [4], they proved the new representation

$$P_\omega(q) := q\omega(q) = \sum_{n\geq 1}\frac{q^n}{(1-q^n)\left(q^{n+1};q\right)_n\left(q^{2n+2};q^2\right)_\infty}. \tag{1.2}$$

Basic combinatorial arguments show that the series (1.2) is the generating function for $p_\omega(n)$, which enumerates the partitions of n in which the odd parts are less than twice the