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On the dynamics of finite temperature trapped Bose gases

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ABSTRACT

The system that describes the dynamics of a Bose–Einstein Condensate (BEC) and the thermal cloud at finite temperature consists of a nonlinear Schrodinger (NLS) and a quantum Boltzmann (QB) equations. In such a system of trapped Bose gases at finite temperature, the QB equation corresponds to the evolution of the density distribution function of the thermal cloud and the NLS is the equation of the condensate. The quantum Boltzmann collision operator in this temperature regime is the sum of two operators C_{12} and C_{22}, which describe collisions of the condensate and the non-condensate atoms and collisions between non-condensate atoms. Above the BEC critical temperature, the system is reduced to an equation containing only a collision operator similar to C_{22}, which possesses a blow-up positive radial solution with respect to the L^∞ norm (cf. [29]). On the other hand, at the very low temperature regime (only a portion of the transition temperature T_{BEC}), the system can be simplified into an equation of C_{12}, with a different (much higher order) transition probability, which has a unique global classical positive radial solution with weighted L^1 norm (cf. [3]). In our model, we first decouple the QB, which contains $C_{12}+C_{22}$, and the NLS equations, then show a global existence and uniqueness result for classical positive radial solutions to the spatially homogeneous kinetic system. Different from the case considered in [29], due to the presence of the BEC, the collision integrals are associated to sophisticated energy manifolds rather than spheres, since the

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particle energy is approximated by the Bogoliubov dispersion law. Moreover, the mass of the full system is not conserved while it is conserved for the case considered in [29]. A new theory is then supplied.

1. Introduction

The study of kinetic equations has a very long history, starting with the classical Boltzmann equation, which provides a description of the dynamics of dilute monoatomic gases (cf. [21–23,42,84]). As an attempt to extend the Boltzmann equation to deal with quantum gases, the Boltzmann–Nordheim (Uehling–Uhlenbeck) equation was introduced [69, 83]. However, the Boltzmann–Nordheim (Uehling–Uhlenbeck) equation fails to describe a Bose gas at temperatures which are close to and below the Bose–Einstein Condensate (BEC) critical temperature, due to the fact that its steady-state solution is a Bose–Einstein distribution in particle energies. Below the critical temperature, many-body effects modify the equilibrium distribution so that this distribution depends on quasiparticle energies. These are accounted for by mean fields which break the unperturbed Hamiltonian $U(1)$ gauge symmetry. Therefore, a new description in terms of quasiparticles is required. Such a quantum kinetic theory was initiated by Kirkpatrick and Dorfman [56,57], based on the rich body of research carried out in the period 1940–67 by Bogoliubov, Lee and Yang, Beliaev, Pitaevskii, Hugenholtz and Pines, Hohenberg and Martin, Gavoret and Nozi‘eres, Kane and Kadanoff and many others. After the production of the first BECs, that later led Cornell, Wieman, and Ketterle to the 2001 Nobel Prize of Physics [4,5,12], there has been an explosion of research on the kinetic theory associated to BECs. Based on Kirkpatrick–Dorfman's works, Zaremba, Nikuni and Griffin successfully formulated a self-consistent Gross–Pitaevskii–Boltzmann model, which is nowadays known as the 'ZNG' theory (cf. [15,89]). Independent of the mentioned authors, Pomeau et al. [70] also proposed a similar model for the kinetics of BECs. Later, Gardinier, Zoller and collaborators derived a Master Quantum Kinetic Equation (MQKE) for BECs, which returns to the ZNG model at the limits, and introduced the terminology "Quantum Kinetic Theory" in the series of papers [33–37,52,53]. The ZNG theory also gave the first quantitative predictions of vortex nucleation at finite temperatures [86]. Many other experiments have also confirmed the validity of the model (cf. [73]). We refer to the review paper [6] for discussions on the condensate growth problem concerning the MQKE model and the books [43,51,72] for more theoretical and experimental justifications of the ZNG model, as well as the tutorial article [73] for an easy introduction. Let us mention that besides the ZNG theory, there have been other works describing the kinetics of BECs as well (see [1,50,55,77,78,80,82], and references therein).

Let us first recall the ZNG model for finite temperature trapped Bose gases, i.e. the temperature T of the gas is below the transition temperature T_{BEC} but above absolute