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The necessity of nowhere equivalence ☆

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A B S T R A C T

We prove some regularity properties (convexity, closedness, compactness and preservation of upper hemicontinuity) for distribution and regular conditional distribution of correspondences under the nowhere equivalence condition. We show the necessity of such a condition for any of these properties to hold. As an application, we demonstrate that the nowhere equivalence condition is satisfied on the underlying agent space if and only if pure-strategy Nash equilibria exist in general large games with any fixed uncountable compact action space.

Contents

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1. Introduction

The theory of correspondences, which has important applications in a variety of areas (including optimization, control theory and mathematical economics), has been studied extensively. However, basic regularity properties on the distribution of correspondences such as convexity, closedness, compactness and preservation of upper hemicontinuity may all fail when the underlying probability space is the Lebesgue unit interval; see, for example, [17] and [37]. These issues were resolved in [37] by considering a class of rich measure spaces, the so-called Loeb measure spaces constructed from the method of nonstandard analysis.[1] It was further shown in [17] that the abstract property of saturation[2] on a probability space is not only sufficient but also necessary for any of these regularity properties to hold.[3]

Theorem 3B.7 of [11, p. 47] by Fajardo and Keisler indicated that a saturated probability space is necessarily rich with measurable sets (also [15, Corollary 4.5] by Hoover and Keisler implicitly) in the sense that any of its nontrivial sub-measure space is not countably generated module null sets. However, standard probability spaces such as complete separable metric spaces with Borel probability measures are only countably generated (thus not saturated). To allow the possibility of working with such standard probability spaces, this paper uses the condition of "nowhere equivalence" to characterize some general results on correspondences and the existence of Nash equilibrium in large games.

As noted in [13], the nowhere equivalence condition was motivated by the fact that various equilibrium properties in economics may require different agents with the same characteristic to choose different actions. Thus, one needs to distinguish the σ-algebra $\mathcal{T}$ in a probability space $(T, \mathcal{T}, \lambda)$ (modeling the space of agents) from the σ-algebra $\mathcal{F}$ generated by the mapping specifying the individual characteristics. The condition captures the idea that for any nontrivial collection D of agents, the σ-algebra $\mathcal{T}$ is *richer*

[1] See [22] and [26] for the construction of Loeb spaces.

[2] As noted in [15], atomless Loeb probability spaces are saturated. For some other applications of Loeb and saturated probability spaces, see, for example, [5], [7], [8], [10], [16], [19], [24], [34], [35], and [36].

[3] When the target space is a Banach space, one can also consider Bochner and Gelfand integration of correspondences. The same kind of regularity properties were shown to hold under Loeb/saturated probability spaces in [30], [38] and [39], while the necessity of saturation for these properties was indicated in [30] and [39]. A related issue on the purification of measure-valued maps on Loeb/saturated probability spaces was considered in [23], [25] and [31] with the necessity of saturation in [25] and [31].