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# Bressoud's conjecture

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ABSTRACT

In 1980, D. M. Bressoud obtained an analytic generalization of the Rogers–Ramanujan–Gordon identities. He then tried to establish a combinatorial interpretation of his identity, which specializes to many well-known Rogers–Ramanujan type identities. He proved that a certain partition identity follows from his identity in a very restrictive case and conjectured that the partition identity holds true in general. In this paper, we prove Bressoud's conjecture for the general case by providing bijective proofs.



## 1. Introduction

The most celebrated identities in the theory of partitions are the Rogers–Ramanujan identities: for $r = 1, 2$,

$$\sum_{n=0}^{\infty} \frac{q^{n^2+(r-1)n}}{(q;q)_n} = \frac{1}{(q^r, q^{5-r}; q^5)_\infty}, \tag{1.1}$$

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where we follow the standard notation

$$\begin{aligned}(a;q)_0 &= 1,\\ (a;q)_n &= (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n \geq 1,\\ (a;q)_\infty &= \lim_{n\to\infty}(a;q)_n, \quad |q|<1,\\ (a_1,\ldots,a_m;q)_\infty &= (a_1;q)_\infty\cdots(a_m;q)_\infty.\end{aligned}$$

The Rogers–Ramanujan identities were discovered by L. J. Rogers [15] in 1894, and S. Ramanujan [13][14, p. 330] rediscovered them. In addition, P. A. MacMahon [12] and I. J. Schur [16] independently discovered the beautiful combinatorial statement of (1.1):

> For each positive integer $n$, and $r = 1, 2$, the number of partitions of $n$ into parts greater than or equal to $r$ with minimal difference 2 is equal to the number of partitions of $n$ into parts congruent to $\pm r \pmod 5$.

Since then, there has been considerable investigation on Rogers–Ramanujan type identities and their combinatorics. In particular, B. Gordon [7] made the first break-through by proving an infinite family of combinatorial generalizations of the Rogers–Ramanujan identities. Throughout this paper, for a partition $\pi$, we write the parts of $\pi$ in weakly increasing order as follows:

$$\pi_s + \pi_{s-1} + \cdots + \pi_1, \quad \text{where} \quad \pi_s \leq \pi_{s-1} \leq \cdots \leq \pi_1.$$

**Theorem 1.1** *(Rogers–Ramanujan–Gordon identities). For $1 \leq r \leq k$, let $A_{k,r}(n)$ be the number of partitions of $n$ into parts $\not\equiv 0, \pm r \pmod{2k+1}$. Let $B_{k,r}(n)$ be the number of partitions $\pi$ of $n$ such that*

$$\pi_i - \pi_{i+k-1} \geq 2,$$

*and at most $r-1$ of the $\pi_i$ are equal to 1. Then for all $n \geq 0$,*

$$A_{k,r}(n) = B_{k,r}(n). \tag{1.2}$$

Andrews [1] provided an analytic proof of Theorem 1.1, and he discovered the generating function for (1.2) in [3]:

$$\sum_{n_1,\ldots,n_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_r+N_{r+1}+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}} = \frac{(q^r, q^{2k+1-r}, q^{2k+1}; q^{2k+1})_\infty}{(q;q)_\infty}, \tag{1.3}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.