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Heisenberg uniqueness pairs corresponding to a finite number of parallel lines

Sayan Bagchi

Stat-Math Unit, Indian Statistical Institute, Kolkata 700102, India

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In this paper, we study the Heisenberg uniqueness pairs corresponding to a finite number of parallel lines Γ. We give a necessary condition and a sufficient condition for a subset Λ of $\mathbb{R}^2$ so that (Γ, Λ) becomes a HUP.

1. Introduction

Let Γ be a curve on $\mathbb{R}^2$ and Λ be a subset of $\mathbb{R}^2$. We call (Γ, Λ) to be a Heisenberg uniqueness pair if

$$\hat{\mu}|_\Lambda = 0 \text{ implies } \mu = 0$$

for every measure μ on $\mathbb{R}^2$ which is supported on Γ and also absolutely continuous with respect to the arc length of the curve. Here $\hat{\mu}$ stands for the Fourier transform of μ defined by

E-mail address: sayansamrat@gmail.com.

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$$\hat{\mu}(\xi,\eta) = \int_{\mathbb{R}^2} e^{\pi i(x\xi + y\eta)} d\mu(x,y)$$

where $\xi, \eta \in \mathbb{R}^2$.

The terminology "Heisenberg uniqueness pair" was first introduced by Hedenmalm and Montes-Rodríguez in [5]. In that paper they also studied the cases where Γ are a hyperbola and two parallel straight lines. After that, Heisenberg uniqueness pairs were studied for some more well-known curves on $\mathbb{R}^2$ by various people. For example, the case of circle and parabola were studied by Nir Lev [8] and Per Sjölin [10] respectively. Please see [2], [6], [9], [3] and [4] for more results in this direction.

In this paper we extend the results of [1] to finite number of parallel lines. However, unlike [1], we are not able to characterize Heisenberg pairs in the case Γ is a set of finitely many parallel lines.

2. Main result

In order to state our result we first set up some notations. Consider a subset E of $\mathbb{R}$ and a point $\xi \in E$. Then we can define the following sets:

For $2 \le l \le n$, $P_l^{E,\xi} = \{\psi : E \to \mathbb{C}$: there is an interval I_ξ around ξ and functions $\varphi_j \in L^1(\mathbb{R})$, $j = 1, 2, ..., l-1$ such that

$$\hat{\varphi_1} + \hat{\varphi_2}\psi + \hat{\varphi_3}\psi^2 + \cdots + \hat{\varphi_{l-1}}\psi^{l-2} + \psi^{l-1} = 0$$

on $I_\xi \cap E\}$. It is worthy to note the particular case $l = 2$, where $P_2^{E,\xi} = \{\psi : E \to \mathbb{C}$: there is an interval I_ξ and functions $\varphi \in L^1(\mathbb{R})$, such that $\hat{\varphi} = \psi$ on $I_\xi \cap E\}$. We have to work with this particular case many times throughout the paper. Note that by Wiener's lemma ([7], page 57), if $\psi \in P_2^{E,\xi}$, then $\frac{1}{\psi} \in P_2^{E,\xi}$.

Let us consider n parallel lines $\Gamma = \mathbb{R} \times \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ where $\alpha_{i+1} - \alpha_i = constant$, $1 \le i \le n-1$. By translation and rescaling discussed in [5], it is enough to consider $\Gamma = \mathbb{R} \times \{0, 1, 2, \cdots, n-1\}$. If μ is a measure such that μ is supported on Γ and absolutely continuous with respect to arc length of Γ, then there exist functions $f_0, f_1, \cdots f_{n-1} \in L^1(\mathbb{R})$, such that

$$\hat{\mu}(\xi,\eta) = \hat{f}_0(\xi) + e^{\pi i\eta}\hat{f}_1(\xi) + e^{2\pi i\eta}\hat{f}_2(\xi) + \cdots + e^{(n-1)\pi i\eta}\hat{f}_{n-1}(\xi). \tag{2.1}$$

Since $\hat{\mu}$ is 2-periodic with respect to the second variable, we may assume Λ to be 2-periodic with respect to the second coordinate.

Let $\Pi(\Lambda)$ be the projection of Λ on x-axis. That is,

$$\Pi(\Lambda) = \{\xi \in \mathbb{R} : (\xi,\eta) \in \Lambda \text{ for some } \eta \in \mathbb{R}\}.$$

Also consider the set,

$$Img(\xi) = \{\eta \in \mathbb{R} : (\xi,\eta) \in \Lambda, 0 \le \eta < 2\}.$$