

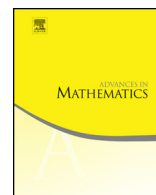


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# Brunn–Minkowski inequalities in product metric measure spaces <sup>☆</sup>

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## ABSTRACT

Given one metric measure space  $X$  satisfying a linear Brunn–Minkowski inequality, and a second one  $Y$  satisfying a Brunn–Minkowski inequality with exponent  $p \geq -1$ , we prove that the product  $X \times Y$  with the standard product distance and measure satisfies a Brunn–Minkowski inequality of order  $1/(1+p^{-1})$  under mild conditions on the measures and the assumption that the distances are strictly intrinsic. The same result holds when we consider restricted classes of sets. We also prove that a linear Brunn–Minkowski inequality is obtained in  $X \times Y$  when  $Y$  satisfies a Prékopa–Leindler inequality.

In particular, we show that the classical Brunn–Minkowski inequality holds for any pair of weakly unconditional sets in  $\mathbb{R}^n$  (i.e., those containing the projection of every point in the set onto every coordinate subspace) when we consider the standard distance and the product measure of  $n$  one-dimensional real measures with positively decreasing densities. This yields an improvement of the class of sets satisfying the Gaussian Brunn–Minkowski inequality.

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Furthermore, associated isoperimetric inequalities as well as recently obtained Brunn–Minkowski’s inequalities are derived from our results.

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## 1. Introduction

The  $n$ -dimensional volume of a set  $M$  in the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  (i.e., its  $n$ -dimensional Lebesgue measure) is denoted by  $\text{vol}(M)$ , or  $\text{vol}_n(M)$  if the distinction of the dimension is useful. The symbol  $\mathbb{B}_n$  stands for the  $n$ -dimensional closed unit ball with respect to the Euclidean norm  $|\cdot|$ .

Relevant families of subsets of Euclidean space used in this work are those of unconditional and weakly unconditional sets: a subset  $A \subset \mathbb{R}^n$  is said to be *unconditional* if for every  $(x_1, \dots, x_n) \in A$  and every  $(\epsilon_1, \dots, \epsilon_n) \in [-1, 1]^n$  one has

$$(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in A.$$

In a similar way, we will say that  $A$  is *weakly unconditional* (see Fig. 1) if for every  $(x_1, \dots, x_n) \in A$  and every  $(\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$  one has

$$(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in A.$$

Weakly unconditional sets are those for which the projection of every point in the set onto any coordinate subspace is again contained in the set. Equivalently, a set  $A$  is weakly unconditional if and only if every non-empty 1-section of  $A$ , through parallel lines to the coordinate axes, contains the origin (identifying the corresponding 1-dimensional affine subspace with its direction; cf. Fig. 1).

Given an arbitrary non-empty set  $B \subset \mathbb{R}^n$ ,  $\overline{B}$  will denote its *weakly unconditional hull* (i.e., the intersection of all weakly unconditional sets containing  $B$ ), which is just the union of  $B$  with every projection of it onto any coordinate subspace. The unconditional hull of  $B$  is defined in a similar way, see Fig. 2.

It is worth mentioning that an unconditional set in  $\mathbb{R}$  is an interval symmetric with respect to the origin, and that a weakly unconditional set in  $\mathbb{R}$  is just a set containing the origin.

Another notion used in this paper is that of *positively decreasing* function. We say that a non-negative function  $f : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  is positively decreasing if the functions  $t \mapsto f(t)$ ,  $t \mapsto f(-t)$  are decreasing (i.e., non-increasing) on  $[0, \infty)$ .

The Minkowski sum of two non-empty sets  $A, B \subset \mathbb{R}^n$  denotes the classical vector addition of them:  $A + B = \{a + b : a \in A, b \in B\}$ . It is natural to wonder about the possibility of relating the volume of the Minkowski sum of two sets in terms of their volumes; this is the statement of the *Brunn–Minkowski inequality*. Indeed, taking

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