Advances in Mathematics 325 (2018) 824–863

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Brunn–Minkowski inequalities in product metric measure spaces ☆

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A R T I C L E   I N F O

Article history:
Received 9 June 2017
Received in revised form 28 November 2017
Accepted 12 December 2017
Available online xxxx
Communicated by Erwin Lutwak

MSC:
primary 52A40, 28A35
secondary 60G15, 52A20

Keywords:
Brunn–Minkowski inequality
Metric measure space
Product space
Gaussian measure
Product measure
Isoperimetric inequality

A B S T R A C T

Given one metric measure space X satisfying a linear Brunn–Minkowski inequality, and a second one Y satisfying a Brunn–Minkowski inequality with exponent $p \geq -1$, we prove that the product $X \times Y$ with the standard product distance and measure satisfies a Brunn–Minkowski inequality of order $1/(1+p^{-1})$ under mild conditions on the measures and the assumption that the distances are strictly intrinsic. The same result holds when we consider restricted classes of sets. We also prove that a linear Brunn–Minkowski inequality is obtained in $X \times Y$ when Y satisfies a Prékopa–Leindler inequality.

In particular, we show that the classical Brunn–Minkowski inequality holds for any pair of weakly unconditional sets in $\mathbb{R}^n$ (i.e., those containing the projection of every point in the set onto every coordinate subspace) when we consider the standard distance and the product measure of n one-dimensional real measures with positively decreasing densities. This yields an improvement of the class of sets satisfying the Gaussian Brunn–Minkowski inequality.

☆ First author is supported by MICINN-FEDER grant MTM2013-48371-C2-1-P and Junta de Andalucía grants FQM-325 and P09-FQM-5088. Second author is partially supported by ICMAT Severo Ochoa project SEV-2015-0554 (MINECO) and by "Programa de Ayudas a Grupos de Excelencia de la Región de Murcia", Fundación Séneca, 19901/GERM/15.

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https://doi.org/10.1016/j.aim.2017.12.010

Furthermore, associated isoperimetric inequalities as well as recently obtained Brunn–Minkowski's inequalities are derived from our results.

1. Introduction

The n-dimensional volume of a set M in the n-dimensional Euclidean space $\mathbb{R}^n$ (i.e., its n-dimensional Lebesgue measure) is denoted by $\mathrm{vol}(M)$, or $\mathrm{vol}_n(M)$ if the distinction of the dimension is useful. The symbol $\mathbb{B}_n$ stands for the n-dimensional closed unit ball with respect to the Euclidean norm $|\cdot|$.

Relevant families of subsets of Euclidean space used in this work are those of unconditional and weakly unconditional sets: a subset $A \subset \mathbb{R}^n$ is said to be *unconditional* if for every $(x_1, \dots, x_n) \in A$ and every $(\epsilon_1, \dots, \epsilon_n) \in [-1, 1]^n$ one has

$$(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in A.$$

In a similar way, we will say that A is *weakly unconditional* (see Fig. 1) if for every $(x_1, \dots, x_n) \in A$ and every $(\epsilon_1, \dots, \epsilon_n) \in \{0, 1\}^n$ one has

$$(\epsilon_1 x_1, \dots, \epsilon_n x_n) \in A.$$

Weakly unconditional sets are those for which the projection of every point in the set onto any coordinate subspace is again contained in the set. Equivalently, a set A is weakly unconditional if and only if every non-empty 1-section of A, through parallel lines to the coordinate axes, contains the origin (identifying the corresponding 1-dimensional affine subspace with its direction; cf. Fig. 1).

Given an arbitrary non-empty set $B \subset \mathbb{R}^n$, $\overline{B}$ will denote its *weakly unconditional hull* (i.e., the intersection of all weakly unconditional sets containing B), which is just the union of B with every projection of it onto any coordinate subspace. The unconditional hull of B is defined in a similar way, see Fig. 2.

It is worth mentioning that an unconditional set in $\mathbb{R}$ is an interval symmetric with respect to the origin, and that a weakly unconditional set in $\mathbb{R}$ is just a set containing the origin.

Another notion used in this paper is that of *positively decreasing* function. We say that a non-negative function $f : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$ is positively decreasing if the functions $t \mapsto f(t)$, $t \mapsto f(-t)$ are decreasing (i.e., non-increasing) on $[0, \infty)$.

The Minkowski sum of two non-empty sets $A, B \subset \mathbb{R}^n$ denotes the classical vector addition of them: $A + B = \{a + b : a \in A, b \in B\}$. It is natural to wonder about the possibility of relating the volume of the Minkowski sum of two sets in terms of their volumes; this is the statement of the *Brunn–Minkowski inequality*. Indeed, taking