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Floer homology and fractional Dehn twists

Matthew Hedden *, Thomas E. Mark

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ABSTRACT

We establish a relationship between Heegaard Floer homology and the fractional Dehn twist coefficient of surface automorphisms. Specifically, we show that the rank of the Heegaard Floer homology of a 3-manifold bounds the absolute value of the fractional Dehn twist coefficient of the monodromy of any of its open book decompositions with connected binding. We prove this by showing that the rank of Floer homology gives bounds for the number of boundary parallel right or left Dehn twists necessary to add to a surface automorphism to guarantee that the associated contact manifold is tight or overtwisted, respectively. By examining branched double covers, we also show that the rank of the Khovanov homology of a link bounds the fractional Dehn twist coefficient of its odd-stranded braid representatives.

1. Introduction

Let S be a compact oriented 2-manifold with a single boundary component, and ϕ a homeomorphism of S fixing its boundary pointwise. The fractional Dehn twist coefficient of ϕ is a rational number $\tau(\phi) \in \mathbb{Q}$ that depends only on the isotopy class of ϕ rel boundary, and can be understood as a measure of the amount of twisting around the boundary effected by ϕ compared to a "canonical"—e.g., pseudo-Anosov—representative

* Corresponding author.
E-mail addresses: mhedden@math.msu.edu (M. Hedden), tmark@virginia.edu (T.E. Mark).

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of its (free) isotopy class. More precisely, consider the image of ϕ under the natural map $\mathrm{Aut}(S, \partial S) \to \mathrm{Aut}(S)$ which drops the requirement that an isotopy fixes the boundary pointwise. In this latter group, ϕ is isotopic to its Nielsen–Thurston representative; that is, there is an isotopy $\Phi : S \times [0,1] \to S$ such that $\Phi_0 = \phi$ and Φ_1 is either periodic, reducible, or pseudo-Anosov.[1] Considering the restriction of Φ to the boundary, we obtain a homeomorphism:

$$\Phi_\partial : \partial S \times [0,1] \to \partial S \times [0,1]$$

defined by $\Phi_\partial(x,t) = (\Phi_t(x), t)$. The fractional Dehn twist coefficient $\tau(\phi)$ can be defined as the winding number of the arc $\Phi(\theta \times [0,1])$ where $\theta \in \partial S$ is a basepoint.[2] This would appear only to associate a real number to ϕ, which could depend on the choice of basepoint and isotopy. The Nielsen–Thurston classification, however, shows that this winding number is a well-defined rational-valued invariant $\tau(\phi) \in \mathbb{Q}$. The definition extends easily to surfaces with several boundary circles, in which case there is a corresponding twist coefficient for each component of the boundary. Here we will be concerned only with the case of connected boundary.

The study of fractional Dehn twist coefficients dates at least from the work of Gabai and Oertel [7] in the context of essential laminations of 3-manifolds, where, with different conventions than those used here, it appeared as the slope of the "degenerate curve" [7, pg. 62]. Honda, Kazez, and Matic [13,14] observed a connection with contact topology through open book decompositions, which has been explored by various authors [3,18, 16]. The following proposition summarizes a few key properties of the fractional Dehn twist coefficient.

Proposition *([21,16]). Let $\tau : \mathrm{Aut}(S, \partial S) \to \mathbb{Q}$ be the fractional Dehn twist coefficient, and let t_∂ denote the mapping class of a right-handed Dehn twist around a curve parallel to ∂S. Then for all $\phi, \psi \in \mathrm{Aut}(S, \partial S)$, we have:*

(1) (Quasimorphism) $|\tau(\phi \circ \psi) - \tau(\phi) - \tau(\psi)| \leqslant 1$.
(2) (Homogeneity) $\tau(\phi^n) = n\tau(\phi)$.
(3) (Boundary Twisting) $\tau(\phi \circ t_\partial) = \tau(\phi) + 1$.

The first two properties easily imply that the fractional Dehn twist is invariant under conjugation (see e.g., [8, Proposition 5.3]), and the third implies that it can be arbitrarily large, either positively or negatively. There are constraints, however, on the possible denominators of $\tau(\phi)$ based on the topology of S; cf. [6, Theorem 8.8], [18, Theorem 4.4], [36].

[1] As in [18], such a map is called reducible only if is not periodic. Moreover, in the reducible case, after an isotopy rel ∂S we get a subsurface of S to which ϕ restricts as a map with periodic or pseudo-Anosov representative: we apply the definition of fractional Dehn twist coefficient to the restriction of ϕ to that subsurface.

[2] $\tau(\phi)$ can be defined without Nielsen–Thurston theory by lifting ϕ to the universal cover and using the translation number of an associated action on a line at infinity [21].