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Norms of random matrices: Local and global problems

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ABSTRACT

Can the behavior of a random matrix be improved by modifying a small fraction of its entries? Consider a random matrix A with i.i.d. entries. We show that the operator norm of A can be reduced to the optimal order $O(\sqrt{n})$ by zeroing out a small submatrix of A if and only if the entries have zero mean and finite variance. Moreover, we obtain an almost optimal dependence between the size of the removed submatrix and the resulting operator norm. Our approach utilizes the cut norm and Grothendieck–Pietsch factorization for matrices, and it combines the methods developed recently by C. Le and R. Vershynin and by E. Rebrova and K. Tikhomirov.

1. Introduction

1.1. Local and global problems

When a certain mathematical or scientific structure fails to meet reasonable expectations, one often wonders: is this a local or global problem? In other words, is the failure

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caused by some small, localized part of the structure, and if so, can this part be identified and repaired? Or, alternatively, is the structure entirely, globally bad? Many results in mathematics can be understood as either local or global statements. For example, not every measurable function $f : \mathbb{R} \to \mathbb{R}$ is continuous, but Lusin's theorem implies that f can always be made continuous by changing its values on a set of arbitrarily small measure. Thus, imposing continuity is a local problem. On the other hand, a continuous function may not be differentiable, and there even exist continuous and nowhere differentiable functions. Thus imposing differentiability may be a global problem. In statistics, the notion of *outliers* – small, pathological subsets of data, the removal of which makes data better – points to local problems.

1.2. Random matrices and their norms

In this paper we ask: is bounding the norm of a random matrix a local or a global problem? To be specific, consider $n \times n$ random matrices A with independent and identically distributed (i.i.d.) entries. The *operator norm* of A is defined by considering A as a linear operator on $\mathbb{R}^n$ equipped with the Euclidean norm $\|\cdot\|_2$, i.e.

$$\|A\| = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

Suppose that the entries of A have zero mean and bounded fourth moment, i.e. $\mathbb{E}\, A_{ij}^4 = O(1)$. Then, as it was shown in [3],

$$\|A\| = (2 + o(1))\sqrt{n}$$

with high probability. Note that the order $\sqrt{n}$ is the best we can generally hope for. Indeed, if the entries of A have unit variance, then the typical magnitude of the Euclidean norm of a row of A is $\sim \sqrt{n}$, and the operator norm of A can not be smaller than that. Moreover, by [4,16] the bounded fourth moment assumption is nearly necessary[2] for the bound

$$\|A\| = O(\sqrt{n}). \tag{1.1}$$

A number of quantitative and more general versions of these bounds are known [15,10,20,5,8,9].

1.3. Main results

Now let us postulate nothing at all about the distribution of the i.i.d. entries of A. It still makes sense to ask: *is enforcing the ideal bound* (1.1) *for random matrices a local*

[2] For almost surely convergence of $\|A\|/\sqrt{n}$, fourth moment is necessary and sufficient [4], while for convergence in probability the weak fourth moment is necessary and sufficient [16].