Advances in Mathematics 324 (2018) 118–147

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Sharp logarithmic estimates for positive dyadic shifts ☆

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ARTICLE INFO

Article history:
Received 30 November 2016
Received in revised form 3 November 2017
Accepted 6 November 2017
Available online xxxx
Communicated by C. Fefferman

MSC:
primary 42B25
secondary 46E30, 60G42

Keywords:
Dyadic shift
Bellman function
Best constants

ABSTRACT

The paper contains the study of sharp logarithmic estimates for positive dyadic shifts $\mathcal{A}$ given on probability spaces (X, μ) equipped with a tree-like structure. For any $K > 0$ we determine the smallest constant $L = L(K)$ such that

$$\int_E |\mathcal{A}f| d\mu \leq K \int_{\mathbb{R}} \Psi(|f|) d\mu + L(K) \cdot \mu(E),$$

where $\Psi(t) = (t+1)\log(t+1) - t$, E is an arbitrary measurable subset of X and f is an integrable function on X. The proof exploits Bellman function method: we extract the above estimate from the existence of an appropriate special function, enjoying certain size and concavity-type conditions. As a corollary, a dual exponential bound is obtained.

☆ Research supported by Narodowe Centrum Nauki (Poland), grant DEC-2014/14/E/ST1/00532.
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https://doi.org/10.1016/j.aim.2017.11.004
0001-8708/

1. Introduction

Let $Q \subset \mathbb{R}^d$ be a given dyadic cube and let $\mathcal{D}(Q)$ stand for the grid of its dyadic subcubes. For a given sequence $\alpha = (\alpha_R)_{R \in \mathcal{D}(Q)}$ of nonnegative numbers, define its Carleson constant by

$$\mathrm{Carl}(\alpha) = \sup_{R \in \mathcal{D}(Q)} \frac{1}{|R|} \sum_{R' \in \mathcal{D}(R)} \alpha_{R'} |R'|,$$

where $|A|$ is the Lebesgue measure of A. For any such sequence, we introduce the associated dyadic shift $\mathcal{A}$ which acts on integrable functions $f : Q \to \mathbb{R}$ by the formula

$$\mathcal{A}f = \sum_{R \in \mathcal{D}(Q)} \alpha_R \langle f \rangle_R \chi_R, \tag{1.1}$$

where $\langle f \rangle_R = \frac{1}{|R|} \int_R f \mathrm{d}\mu$ is the average of f over R.

The class of positive dyadic shifts arose in the works of A. Lerner during his study of the A_2 theorem. Let us discuss this issue in a little more detailed manner. Assume that T is a Calderón–Zygmund operator on $\mathbb{R}^d$ and let $w : \mathbb{R}^d \to (0, \infty)$ be a weight satisfying Muckenhoupt's condition A_2. The so-called A_2 conjecture asked for the linear dependence of the norm $||T||_{L^2(w) \to L^2(w)}$ on $[w]_{A_2}$, the A_2 characteristic of w:

$$||Tf||_{L^2(w)} \leq C(T, d)[w]_{A_2} ||f||_{L^2(w)}.$$

This question has gained a lot of interest in the recent literature (see e.g. [1,4,9,15, 17–20,25]) and was finally answered in the positive by Hÿtonen [6], with the use of clever representation of T as an average of good dyadic shifts. Later, Lerner [11] provided a simpler proof of the A_2 theorem, which avoided the use of most of the complicated techniques in [6]. The idea was to exploit a general pointwise estimate for T in terms of positive dyadic operators, proven in [10]. This allowed to reduce the A_2 problem to a weighted result for the positive dyadic shifts, which had been already shown before in [8] (consult also [4] and [5]). The aforementioned pointwise bound states that for every dyadic cube Q,

$$|Tf(x)| \lesssim \sum_{m=0}^{\infty} 2^{-\delta m} \mathcal{A}_{\mathcal{S}}^m |f|(x) \qquad \text{for a.e. } x \in Q, \tag{1.2}$$

where $\delta > 0$ depends on the operator T, $\mathcal{S}$ is a collection of dyadic cubes which depends on f, T and m, and $\mathcal{A}_{\mathcal{S}}^m$ are positive dyadic operators defined by

$$\mathcal{A}_{\mathcal{S}}^m f(x) = \sum_{Q \in \mathcal{S}} \langle f \rangle_{Q^{(m)}} \chi_Q(x),$$