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Badly approximable points on planar curves and winning

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ABSTRACT

For any $i, j > 0$ with $i + j = 1$, let $\mathbf{Bad}(i, j)$ denote the set of points $(x, y) \in \mathbb{R}^2$ such that $\max\{\|qx\|^{1/i}, \|qy\|^{1/j}\} > c/q$ for some positive constant $c = c(x, y)$ and all $q \in \mathbb{N}$. We show that $\mathbf{Bad}(i, j) \cap \mathcal{C}$ is winning in the sense of Schmidt games for a large class of planar curves $\mathcal{C}$, namely, everywhere non-degenerate planar curves and straight lines satisfying a natural Diophantine condition. This strengthens recent results solving a problem of Davenport from the sixties. In short, within the context of Davenport's problem, the winning statement is best possible. Furthermore, we obtain the inhomogeneous generalisations of the winning results for planar curves and lines and also show that the inhomogeneous form of $\mathbf{Bad}(i, j)$ is winning for two dimensional Schmidt games.

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1. Introduction

A real number x is said to be *badly approximable* if there exists a positive constant $c(x)$ such that

$$\|qx\| \; > \; c(x)\; q^{-1} \quad \forall\, q \in \mathbb{N}\, .$$

Here and throughout $\|\cdot\|$ denotes the distance of a real number to the nearest integer. It is well-known that the set **Bad** of badly approximable numbers is of Lebesgue measure zero but of maximal Hausdorff dimension; i.e. $\dim \mathbf{Bad} = 1$. In higher dimensions there are various natural generalisations of **Bad**. Restricting our attention to the plane $\mathbb{R}^2$, given a pair of real numbers i and j such that

$$0 < i, j < 1 \quad \text{and} \quad i + j = 1\,, \tag{1.1}$$

a point $(x, y) \in \mathbb{R}^2$ is said to be (i, j)-*badly approximable* if there exists a positive constant $c(x, y)$ such that

$$\max\{\, \|qx\|^{\frac{1}{i}}\,,\; \|qy\|^{\frac{1}{j}}\,\} \; > \; c(x,y)\; q^{-1} \quad \forall\, q \in \mathbb{N}\, .$$

Denote by $\mathbf{Bad}(i, j)$ the set of (i, j)-badly approximable points in $\mathbb{R}^2$. In the case $i = j = 1/2$, the set under consideration is the standard set of simultaneously badly approximable points. It easily follows from classical results in the theory of metric Diophantine approximation that $\mathbf{Bad}(i, j)$ is of (two-dimensional) Lebesgue measure zero. Regarding dimension, it was shown by Schmidt [14] in the vintage year of 1966 that $\dim \mathbf{Bad}(\frac{1}{2}, \frac{1}{2}) = 2$. In fact, Schmidt proved the significantly stronger statement that $\mathbf{Bad}(\frac{1}{2}, \frac{1}{2})$ is winning in the sense of his now famous (α, β)-games – see §2.1. Almost forty years later it was proved in [12] that $\dim \mathbf{Bad}(i, j) = 2$ and just recently the first author in [2] has shown that $\mathbf{Bad}(i, j)$ is in fact winning. The latter implies that any countable intersection of $\mathbf{Bad}(i, j)$ sets is of full dimension and thus provides a clean and direct proof of Schmidt's Conjecture – see also [1,3].

Now let $\mathcal{C}$ be a planar curve. Without loss of generality, we assume that $\mathcal{C}$ is given as a graph

$$\mathcal{C}_f := \{(x, f(x)) : x \in I\}$$

for some function f defined on an interval $I \subset \mathbb{R}$. Throughout we will assume that $f \in C^{(2)}(I)$, a condition that conveniently allows us to define the curvature. Motivated by a problem of Davenport [9, p. 52] from the sixties, the following statement regarding the intersection of $\mathbf{Bad}(i, j)$ sets with any curve $\mathcal{C}$ that is not a straight line segment has recently been established [4,5].

Theorem A. *Let (i_t, j_t) be a countable number of pairs of real numbers satisfying (1.1) and suppose that*