Advances in Mathematics 324 (2018) 203–240

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Cells with many facets in a Poisson hyperplane tessellation

Gilles Bonnet [a,*,1], Pierre Calka [b,*], Matthias Reitzner [c,*]

[a] *Faculty of Mathematics, Ruhr University Bochum, Universitätstr. 150, 44780 Bochum, Germany*
[b] *Laboratoire de Mathématiques Raphaël Salem, Université de Rouen, Avenue de l'Université, BP. 12, Technopôle du Madrillet, F76801 Saint-Etienne-duRouvray, France*
[c] *Institut für Mathematik, Universität Osnabrück, Albrechtstr. 28a, 49076 Osnabrück, Germany*

ARTICLE INFO

Article history:
Received 15 November 2016
Accepted 14 November 2017
Available online xxxx
Communicated by Erwin Lutwak

MSC:
60D05
52A22

Keywords:
Poisson hyperplane tessellation
Random polytopes
Typical cell
Directional distribution
Complementary Theorem
D.G. Kendall's problem

ABSTRACT

Let Z be the typical cell of a stationary Poisson hyperplane tessellation in $\mathbb{R}^d$. The distribution of the number of facets $f(Z)$ of the typical cell is investigated. It is shown, that under a *well-spread* condition on the directional distribution, the quantity $n^{\frac{2}{d-1}} \sqrt[n]{\mathbb{P}(f(Z)=n)}$ is bounded from above and from below. When $f(Z)$ is large, the isoperimetric ratio of Z is bounded away from zero with high probability.

These results rely on one hand on the Complementary Theorem which provides a precise decomposition of the distribution of Z and on the other hand on several geometric estimates related to the approximation of polytopes by polytopes with fewer facets.

From the asymptotics of the distribution of $f(Z)$, tail estimates for the so-called Φ content of Z are derived as well as results on the conditional distribution of Z when its Φ content is large.

* Corresponding authors.
E-mail addresses: gilles.bonnet@rub.de (G. Bonnet), pierre.calka@univ-rouen.fr (P. Calka), matthias.reitzner@uni-osnabrueck.de (M. Reitzner).

[1] Supported by the DFG (project GK 1916).

https://doi.org/10.1016/j.aim.2017.11.016

1. Introduction

One of the classical models in stochastic geometry to generate a random mosaic is the construction via a Poisson hyperplane process. A Poisson hyperplane process consists of countably many random hyperplanes in $\mathbb{R}^d$ chosen in such a way, that their distribution is translation invariant, the distribution of the direction of the hyperplanes follows a directional distribution φ, i.e. an even probability measure on the unit sphere which is not concentrated on some great circle, and the number of hyperplanes hitting an arbitrary convex set K is Poisson distributed.

Such a Poisson hyperplane process tessellates $\mathbb{R}^d$ into countably many convex polytopes, the tiles of the mosaic, see e.g. [23, Theorem 10.3.2]. The distribution of a tile chosen at random is the distribution of the so-called *typical cell* Z, a random polytope.

The typical cell has been investigated intensively in the past decades, numerous papers have been dedicated to describe quantities associated with this cell, for example volume, surface area, mean width, number of facets, etc. The expected number of facets $f(Z)$ of the typical cell and the expected volume $V_d(Z)$ are known, see e.g. the first works due to Miles [18,19] and Matheron [16] as well as Chapter 10 from the seminal book of Schneider and Weil [23] and the survey [3].

But in almost all cases the distribution of these quantities is out of reach, and even good approximations are extremely difficult and unknown so far. Our main theorem fills this gap for the number of facets of Z, giving precise asymptotics for the tails of the distribution.

Theorem 1.1. *There exists a constant $c_1 > 0$, depending on φ, such that for $n \geq d+1$,*

$$\mathbb{P}(f(Z) = n) < c_1^n \, n^{-\frac{2n}{d-1}}.$$

Furthermore, there exists an integer n_φ such that $\mathbb{P}(f(Z) = n)$ is either vanishing or strictly decreasing for $n \geq n_\varphi$.

Here and in the sequel, c_i will denote a positive constant which depends on dimension d. It will be specified when it depends on φ or another parameter.

It is clear that in general there is no matching lower bound, for example if the directions of the hyperplane process are concentrated on a finite set. We prove that, if the directional distribution satisfies a mild condition, we have lower bounds of the same order in n as the upper bound above. In the following, we call φ *well spread* if there exists a cap on the unit sphere where φ is bounded from below by a multiple of the spherical Lebesgue measure.

Theorem 1.2. *Assume that φ is well spread. Then there exists a constant $c_2 > 0$, depending on φ, such that for $n \geq d+1$,*

$$\mathbb{P}(f(Z) = n) > c_2^n \, n^{-\frac{2n}{d-1}}.$$