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On the gonality and the slope of a fibered surface

Xin Lu [a,*,1,2], Kang Zuo [b,2]

[a] *Department of Mathematics, Shanghai Key Laboratory of PMMP, East China Normal University, Dongchuan RD 500, Shanghai 200241, PR China*
[b] *Institut für Mathematatik, Universität Mainz, Mainz, 55099, Germany*

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Let $f : X \to B$ be a locally non-trivial relatively minimal fibration of curves of genus $g \geq 2$. We obtain a lower bound of the slope $\lambda(f)$ increasing with the gonality of the general fiber of f. In particular, we show that $\lambda(f) \geq 4$ provided that f is non-hyperelliptic and $g \geq 16$.

* Corresponding author.
E-mail addresses: x.lu@uni-mainz.de (X. Lu), zuok@uni-mainz.de (K. Zuo).
[1] *Current address:* Institut für Mathematik, Universität Mainz, Mainz, 5509, Germany.
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1. Introduction

In this paper, we always work over the complex number $\mathbb{C}$. A fibered surface, or simply a fibration, is a surjective proper morphism $f : X \to B$ from a non-singular projective surface X onto a non-singular projective curve B with connected fibers. A general fiber of f is a smooth curve of genus g, which will be assumed to be at least 2 in the paper. If the general fiber of f is a hyperelliptic curve, then we call f a hyperelliptic fibration. We always assume that f is relatively minimal, i.e., there is no exceptional curve contained in the fibers of f. It is called smooth if all its fibers are smooth, isotrivial if all its smooth fibers are isomorphic to each other, and locally trivial if it is both smooth and isotrivial.

Let ω_X (resp. ω_B) be the canonical sheaf of X (resp. B), and $\omega_{X/B} = \omega_X \otimes f^*\omega_B^\vee$ the relative canonical sheaf of f. The relative minimality of f implies that $\omega_{X/B}$ is numerical effective, i.e., $\omega_{X/B} \cdot C \geq 0$ for any curve $C \subseteq X$. Let $\chi(\mathcal{O}_X)$ be the Euler characteristic of the structure sheaf $\mathcal{O}_X$, and $\chi_{\mathrm{top}}(X)$ the topological Euler characteristic of X. Then we consider the following relative invariants of f:

$$\begin{cases} \omega_{X/B}^2 = \omega_X^2 - 8(g-1)\big(g(B)-1\big), \\ \deg f_*\omega_{X/B} = \chi(\mathcal{O}_X) - (g-1)\big(g(B)-1\big), \\ \delta_f = \chi_{\mathrm{top}}(X) - 4(g-1)\big(g(B)-1\big). \end{cases}$$

They satisfy the following properties:

$$12\deg f_*\omega_{X/B} = \omega_{X/B}^2 + \delta_f.$$
$$\delta_f \geq 0;\ \text{moreover, } \delta_f = 0 \text{ iff } f \text{ is smooth.}$$
$$\deg f_*\omega_{X/B} \geq 0;\ \text{moreover, } \deg f_*\omega_{X/B} = 0 \text{ iff } f \text{ is locally trivial.}$$

If f is not locally trivial, the slope of f is defined to be

$$\lambda(f) = \frac{\omega_{X/B}^2}{\deg f_*\omega_{X/B}}.$$

It follows immediately that $0 < \lambda(f) \leq 12$. It turns out that the slope $\lambda(f)$ is sensible to a lot of geometric properties, both of the fibers of f and of the surface X itself (cf. [4]). We are mainly concerned about a lower bound of the slope. The main known result is the slope inequality:

$$\text{If } f \text{ is not locally trivial, then } \lambda(f) \geq \frac{4(g-1)}{g}. \tag{1.1}$$

It was first proved by Horikawa and Persson for hyperelliptic fibrations. Xiao gave a proof for general fibrations [31], and independently, Cornalba and Harris proved it for semi-stable fibrations [12] (their method was generalized to the general case later by