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Finite orbits in multivalued maps and Bernoulli convolutions ☆

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ABSTRACT

Bernoulli convolutions are certain measures on the unit interval depending on a parameter β between 1 and 2. In spite of their simple definition, they are not yet well understood. We study their two-dimensional density which exists by a theorem of Solomyak. To each Bernoulli convolution, there is an interval D called the overlap region, and a map which assigns two values to each point of D and one value to all other points of $[0, 1]$. There are two types of finite orbits of these multivalued maps which correspond to zeros and potential singularities of the density, respectively.

Orbits which do not meet D belong to an ordinary map called β-transformation and exist for all $\beta > 1.6182$. They were studied by Erdös, Jóo, Komornik, Sidorov, de Vries and others as points with unique addresses, and by Jordan, Shmerkin and Solomyak as points with maximal local dimension. In the two-dimensional view, these orbits form address curves related to the Milnor–Thurston itineraries in one-dimensional dynamics. The curves depend smoothly on the parameter and represent quantiles of all corresponding Bernoulli convolutions.

Finite orbits which intersect D have a network-like structure and can exist only at Perron parameters β. Their points are intersections of extended address curves, and can have finite or countable number of addresses, as found by Sidorov. For an uncountable number of parameters, the central point $\frac{1}{2}$ has only two addresses. The intersection of periodic address curves

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can lead to singularities of the measures. We give examples which are not Pisot or Salem parameters.

1. Introduction

This paper studies two related subjects: the occurrence of network-like orbits in multivalued maps, and the parametric family of Bernoulli convolutions, BCs for short. The orbit of a point x in a dynamical system can be visualized as a path of arrows leading from $f^k(x)$ to $f^{k+1}(x)$ for $k = 0, 1, 2, \ldots$ The orbit becomes finite if x fulfills the equation $f^{k+p}(x) = f^k(x)$ for some k and p, so that the path turns into a cycle. For a multivalued map, the orbit of x is represented by a branching tree. Such orbit can only be finite if all branches lead back to lower levels of the tree, which is usually expressed by several equations, determining not only the point x but also the mapping f, at least to some extent. Nevertheless, network-like orbits appear in simple multivalued maps of an interval, in particular those which define BCs.

Bernoulli convolutions are the simplest examples of self-similar measures with overlaps. They have been studied as examples in real analysis since the 1930s. Given a number β between 1 and 2, let $t = 1/\beta$ and consider the two linear functions

$$g_0 : [0, t] \to [0, 1],\ g_0(x) = \beta x \quad \text{and} \quad g_1 : [1 - t, 1] \to [0, 1],\ g_1(x) = \beta x + 1 - \beta\,, \quad (1)$$

as indicated in Fig. 1. The BC with parameter β is the unique probability measure ν on $[0, 1]$ which fulfills

$$\nu(A) = \frac{1}{2}\nu(g_0(A)) + \frac{1}{2}\nu(g_1(A)) \quad \text{for all Borel sets } A \subset [0, 1]. \quad (2)$$

When A is a subset of $[0, 1 - t]$ or $[t, 1]$, only one term will appear on the right of (2) since $g_1(A)$ or $g_0(A)$ is empty. The map $G(x) = \{g_0(x), g_1(x)\}$ is multivalued only for $x \in [1 - t, t] = D$, the so-called overlap region. Basic facts on BCs can be found in [8,42,51] and in Section 6. Definitions of Pisot and Garsia numbers are given in Section 4.

Since Erdös [17] proved in 1939 that BCs for Pisot numbers β are singular measures, much work was done to determine those β for which ν is absolutely continuous. There are detailed studies of BCs of Pisot numbers [4,20,22,27,36] and other algebraic numbers [21,23,24,54]. Singularity of ν is still only known for the countable set of Pisot numbers, and regularity of specific Bernoulli convolutions could be shown only for the countable set of Garsia numbers [24], not even for any particular rational number β. However, Solomyak [50] proved in 1995 that for Lebesgue almost all β the measure ν has a density function, even an L^2 density function as shown by Peres and Solomyak [41]. In other words, if we take a random number β between 1 and 2, then ν will have a density function with