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Modular characteristic classes for representations over finite fields

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ABSTRACT

The cohomology of the degree-n general linear group over a finite field of characteristic p, with coefficients also in characteristic p, remains poorly understood. For example, the lowest degree previously known to contain nontrivial elements is exponential in n. In this paper, we introduce a new system of characteristic classes for representations over finite fields, and use it to construct a wealth of explicit nontrivial elements in these cohomology groups. In particular we obtain nontrivial elements in degrees linear in n. We also construct nontrivial elements in the mod p homology and cohomology of the automorphism groups of free groups, and the general linear groups over the integers. These elements reside in the unstable range where the homology and cohomology remain mysterious.

1. Introduction

We introduce a new system of modular characteristic classes for representations of groups over finite fields, and use it to construct explicit non-trivial elements in the modular cohomology of the general linear groups over finite fields. The cohomology groups $H^*(GL_N\mathbb{F}_{p^r};\mathbb{F})$ were computed by Quillen [32] in the case where $\mathbb{F}$ is a field of

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characteristic different from p, but he remarked that determining them in the modular case where the characteristic of $\mathbb{F}$ is p "seems to be a difficult problem once $N \geq 3$" [32, p. 578]. Indeed, the modular cohomology has since resisted computation for four decades. Complete calculations exist only for $N \leq 4$ [3,37,36,1]. Much attention has focused on the case where N is small compared to p, e.g. [6–8,34].

To our knowledge, when $N > \max\{p, 4\}$, the only previously constructed nonzero elements of $H^*(GL_N\mathbb{F}_{p^r};\mathbb{F}_p)$ are those due to Milgram and Priddy [30], in the case $r = 1$. These reside in exponentially high degree: at least p^{N-2}. On the other hand, the cohomology is known to vanish in degrees less than $N/2$, by the stability theorem of Maazen [27] together with Quillen's computation [32] that the stable limit is zero. This leaves a large degree gap where it was not known whether the cohomology groups are nontrivial. We narrow this gap considerably by providing nonzero classes in degrees linear in N. We obtain:

Theorem 1 *(see Theorem 40). Let $N \geq 2$, and let n be the natural number satisfying*

$$p^{n-1} < N \leq p^n.$$

Then

$$H^*(GL_N\mathbb{F}_{p^r};\mathbb{F}_p)$$

has a nonzero element in degree $r(2p^n - 2p^{n-1} - 1)$. Moreover, it has a non-nilpotent element in degree $2r(p^n - 1)$ if p is odd and in degree $r(2^n - 1)$ if $p = 2$. □

Notice that the degrees in the theorem grow linearly with N: if d is any of the degrees mentioned in the theorem, then

$$r(N-1) \leq d < 2r(pN-1).$$

In the case $r = 1$, we obtain stronger results, for instance:

Theorem 2 *(see Theorem 42). For all $N \geq 2$,*

$$H^*(GL_N\mathbb{F}_2;\mathbb{F}_2)$$

has a non-nilpotent element of degree d for every d with at least $\lceil \log_2 N \rceil$ ones in its binary expansion. □

Our characteristic classes are defined for representations of dimension $N \geq 2$ over the finite field $\mathbb{F}_{p^r}$, and they are modular in the sense that they take values in group cohomology with coefficients in a field $\mathbb{F}$ of characteristic p. Thus they are interesting even for p-groups. The family of characteristic classes is parametrized by the cohomology of $GL_2\mathbb{F}_{p^r}$. We will show that many classes in this family are nonzero by finding