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Filtrations, 1-parameter subgroups, and rational injectivity

Eric M. Friedlander [1]

Department of Mathematics, University of Southern California, Los Angeles, CA, United States

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ABSTRACT

We investigate rational G-modules M for a linear algebraic group G over an algebraically closed field k of characteristic $p > 0$ using filtrations by sub-coalgebras of the coordinate algebra $k[G]$ of G. Even in the special case of the additive group $\mathbb{G}_a$, interesting structures and examples are revealed. The "degree" filtration we consider for unipotent algebraic groups leads to a "filtration by exponential degree" applicable to rational G modules for any linear algebraic group G of exponential type; this filtration is defined in terms of 1-parameter subgroups and is related to support varieties introduced recently by the author for such rational G-modules. We formulate in terms of this filtration a necessary and sufficient condition for rational injectivity for rational G-modules. Our investigation leads to the consideration of two new classes of rational G-modules: those that are "mock injective" and those that are "mock trivial".

E-mail addresses: ericmf@usc.edu, eric@math.northwestern.edu.

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0. Introduction

Beginning with the very special case of the additive group $\mathbb{G}_a$, we consider the filtration by degree on rational $\mathbb{G}_a$-modules which enables us to better understand the intriguing category ($\mathbb{G}_a$-Mod) of rational $\mathbb{G}_a$-modules. This filtration leads to a similarly defined filtration by degree on rational U_N-modules, where $U_N \subset GL_N$ is the closed subgroup of strictly upper triangular matrices, and determines a filtration on rational U-modules for a closed linear subgroup $U \subset U_N$. We then initiate the study of a less evident filtration on rational G-modules for G a linear algebraic group of exponential type. For rational U_N-modules for the unipotent algebraic group U_N, this filtration of exponential degree is a comparable to the more elementary filtration by degree we first consider. Throughout, we fix an algebraic closed field k of characteristic $p > 0$ and consider (smooth) linear algebraic groups over k together with their rational actions on k-vector spaces.

In some sense, this paper is a sequel to the author's recent paper [4] in which a theory of support varieties $M \mapsto V(G)_M$ was constructed for rational G-modules. The construction of the filtration by exponential degree $\{M_{[0]} \subset M_{[1]} \subset \cdots \subset M\}$ uses restrictions of M to 1-parameter subgroups $\mathbb{G}_a \to G$, and thus is based upon actions of G on M at p-unipotent elements of G. The role of 1-parameter subgroups to study rational G-modules was introduced in [3]; the property of p-unipotent degree introduced in [3, 2.5] is the precursor to our filtration by exponential degree. The origins of this approach to filtrations lie in considerations of support varieties for modules for infinitesimal group schemes, varieties which are defined in terms of p-nilpotent actions on such modules.

The basic theme of this paper is to investigate rational G-modules through their restrictions via 1-parameter subgroups $\mathbb{G}_a \to G$. Whereas the support variety construction $M \mapsto V(G)_M$ is defined in terms of restrictions of M to a p-nilpotent operator associated to each 1-parameter subgroup of G, our present approach involves the full information of the restriction of M to all 1-parameter subgroups by using filtrations on the category of rational G-modules. We give a necessary and sufficient condition for rational injectivity of a rational G-module, something we have not succeeded in doing using support varieties. Moreover, these filtrations enable us to formulate and study the classes of mock injective modules (those whose restrictions to every Frobenius kernel are injective) and of mock trivial modules (those for which actions at 1-parameter subgroups are trivial). These modules are somewhat elusive to construct, but can be shown to exist in great numbers and have interesting properties.

Perhaps the groups of most interest are reductive groups, especially simple groups of classical type. For such groups G, it is instructive to compare the approach in this paper and in [4] with traditional considerations of weights for the action of a maximal torus $T \subset G$ on a rational G-module. Whereas consideration of weights for T are highly suitable in classifying irreducible rational G-modules, the action at p-unipotent elements has the potential of recognizing extensions of such modules. Although our filtration by exponential degree involves actions at p-unipotent elements of G, Example 3.14 shows that a bound on the T-weights for a rational G-module M determines a bound on the