Advances in Mathematics 323 (2018) 193–225

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Periodic distributions and periodic elements in modulation spaces

Joachim Toft *, Elmira Nabizadeh

Department of Mathematics, Linnæus University, Växjö, Sweden

ARTICLE INFO

Article history:
Received 10 February 2017
Received in revised form 7 October 2017
Accepted 16 October 2017
Available online 7 November 2017
Communicated by Kenneth Falconer

MSC:
primary 42B05, 42B35, 46F99, 46Exx
secondary 46B40

Keywords:
Fourier coefficients
Periodic
Ultradistribution
Gevrey classes
Gelfand–Shilov spaces

ABSTRACT

We characterize periodic elements in Gevrey classes, Gelfand–Shilov distribution spaces and modulation spaces, in terms of estimates of involved Fourier coefficients, and by estimates of their short-time Fourier transforms. If $q \in [1, \infty)$, ω is a suitable weight and $(\mathcal{E}_0^E)'$ is the set of all E-periodic elements, then we prove that the dual of $M^{\infty,q}_{(\omega)} \cap (\mathcal{E}_0^E)'$ equals $M^{\infty,q'}_{(1/\omega)} \cap (\mathcal{E}_0^E)'$ by suitable extensions of Bessel's identity.

0. Introduction

A fundamental issue in analysis concerns periodicity. For example, several problems in the theory of partial differential equations and in signal processing involve periodic

* Corresponding author.
E-mail addresses: joachim.toft@lnu.se (J. Toft), elmira.nabizadeh.extern@lnu.se (E. Nabizadeh).

https://doi.org/10.1016/j.aim.2017.10.040
0001-8708/

functions and distributions. In such situations it is in general possible to discretize the problems by means of Fourier series expansions of these functions and distributions.

We recall that if f is a smooth 1-periodic function on $\mathbf{R}^d$, then f is equal to its Fourier series

$$\sum_{\alpha \in \mathbf{Z}^d} c(\alpha) e^{2\pi i \langle \, \cdot \, , \alpha \rangle}, \tag{0.1}$$

where the Fourier coefficients $c(\alpha)$ can be evaluated by the formula

$$c(f,\alpha) = c(\alpha) = \int\limits_{[0,1]^d} f(x) e^{-2\pi i \langle x, \alpha \rangle} \, dx.$$

(Our investigations later on involve functions and distributions with more general periodics. See also [18], and Sections 1 and 2 for notations.) By the smoothness of f it follows that for every $N \ge 0$, there is a constant $C_N \ge 0$ such that

$$|c(\alpha)| \le C_N \langle \alpha \rangle^{-N}, \tag{0.2}$$

and it follows from Weierstrass theorem that the series (0.1) is uniformly convergent (cf. e.g. [18, Section 7.2]). Here $\langle x \rangle = (1 + |x|)$.

Assume instead that f is a 1-periodic distribution on $\mathbf{R}^d$, and let ϕ be compactly supported and smooth on $\mathbf{R}^d$ such that

$$\sum_{k \in \mathbf{Z}^d} \phi(\, \cdot \, - k) = 1. \tag{0.3}$$

Then f is a tempered distribution and is still equal to its Fourier series (0.1) in distribution sense. The Fourier coefficients for f are uniquely defined and can be computed by

$$c(f,\alpha) = c(\alpha) = \langle f, \phi e^{-i \langle \, \cdot \, , \alpha \rangle} \rangle, \tag{0.4}$$

and satisfy

$$|c(\alpha)| \le C \langle \alpha \rangle^{N}, \tag{0.5}$$

for some constants C and N which only depend on f. In particular, $c(f,\alpha)$ is independent of the compactly supported smooth ϕ in (0.3) and (0.4). (Cf. e.g. [18, Section 7.2]. See also [32] for an early approach to formal Fourier series expansions.)

The conditions (0.2) and (0.5) are not only necessary but also sufficient for a formal Fourier series expansion (0.1) being smooth respectively a tempered distribution. Hence, by a unique extension of Parseval's identity