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Canonical tilting relative generators

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A B S T R A C T

Given a relatively projective birational morphism $f: X \to Y$ of smooth algebraic spaces with dimension of fibers bounded by 1, we construct tilting relative (over Y) generators $T_{X,f}$ and $S_{X,f}$ in $\mathcal{D}^b(X)$. We develop a piece of general theory of strict admissible lattice filtrations in triangulated categories and show that $\mathcal{D}^b(X)$ has such a filtration $\mathcal{L}$ where the lattice is the set of all birational decompositions $f: X \xrightarrow{g} Z \xrightarrow{h} Y$ with smooth Z. The t-structures related to $T_{X,f}$ and $S_{X,f}$ are proved to be glued via filtrations left and right dual to $\mathcal{L}$. We realise all such Z as the fine moduli spaces of simple quotients of $\mathcal{O}_X$ in the heart of the t-structure for which $S_{X,g}$ is a relative projective generator over Y. This implements the program of interpreting relevant smooth contractions of X in terms of a suitable system of t-structures on $\mathcal{D}^b(X)$.

Contents

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Introduction

This paper is devoted to the categorical study of relatively projective birational morphisms $f\colon X \to Y$ between smooth algebraic spaces with the dimension of fibres bounded by 1. According to a theorem of V. Danilov such a morphism has a decomposition into a sequence of blow-ups with smooth centers of codimension 2. Our goal is to find a categorical interpretation for f and for all possible intermediate contractions in terms of transformations of t-structures in the bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves on X.

Recall that T. Bridgeland, in his approach to proving the derived flop conjecture (see [9]) in dimension 3, introduced in [12] a series of t-structures in $\mathcal{D}^b(X)$ related to a birational morphism $f\colon X \to Y$ of projective varieties with fibers of dimension bounded by 1. The t-structures, with hearts ${}^p\mathrm{Per}(X/Y)$, depended on an integer parameter $p \in \mathbb{Z}$. Under the assumption that f was a flopping contraction, he used these t-structures to define the flopped variety as a moduli space of so-called *point objects* in ${}^{-1}\mathrm{Per}(X/Y)$.

In our setting of *divisorial contractions* instead of flopping contractions, we construct a system of t-structures with nice properties and interpret all possible intermediate smooth contractions between X and Y as the fine moduli spaces of simple quotients of $\mathcal{O}_X$ in the hearts of those t-structures.

We study the partially ordered set $\mathrm{Dec}(f)$ of all decompositions for f into two birational morphisms with a smooth intermediate space. We prove that it is a distributive lattice and identify it with the lattice of lower ideals in a poset $\mathrm{Conn}(f)$, which is a subposet in $\mathrm{Dec}(f)$ (see Corollary 2.15). We provide with various descriptions of $\mathrm{Conn}(f)$