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Green tensor of the Stokes system and asymptotics of stationary Navier–Stokes flows in the half space

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We derive refined estimates of the Green tensor of the stationary Stokes system in the half space. We then investigate the spatial asymptotics of stationary solutions of the incompressible Navier–Stokes equations in the half space. We also discuss the asymptotics of fast decaying flows in the whole space and exterior domains. In the Appendix we consider axisymmetric self-similar solutions.

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1. Introduction

We are concerned with the Stokes system in the n-dimensional half space $\mathbb{R}^n_+$, $n \geq 2$,

$$\begin{cases} -\Delta u + \nabla q = f + \nabla \cdot F, & \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^n_+, \\ \qquad u = 0 & \qquad\qquad \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{S}$$

or of the Navier–Stokes equations

$$\begin{cases} -\Delta u + (u \cdot \nabla)u + \nabla p = f + \nabla \cdot F, & \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^n_+, \\ \qquad u = 0 & \qquad\qquad \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{NS}$$

Above $u = (u_i)_{i=1}^n : \mathbb{R}^n_+ \to \mathbb{R}^n$ is the velocity field, $p : \mathbb{R}^n_+ \to \mathbb{R}$ is the pressure, and $(f + \nabla \cdot F)_i = f_i + \partial_j F_{ji}$ is the given force. We denote

$$\mathbb{R}^n_+ = \left\{ x = (x', x_n) : \ x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, \ x_n > 0 \right\}, \tag{1.1}$$

with boundary $\Sigma = \partial\mathbb{R}^n_+ = \left\{ x = (x', x_n) : \ x' \in \mathbb{R}^{n-1}, \ x_n = 0 \right\}$. Denote

$$x^* = (x', -x_n) \quad \text{if} \quad x = (x', x_n). \tag{1.2}$$

The purpose of this paper is to study the asymptotic behavior of the Navier–Stokes flows for small forces. To this end, we also derive pointwise estimates of the Green tensor for the Stokes system (S). Our linear results are valid for dimension $n \geq 2$, while our nonlinear results are mostly for $n \geq 3$.

1.1. Background and motivation

As shown by Lorentz [11] (see also [16,5], §2.1), the fundamental solution $\{U_{ij}(x)\}_{i,j=1,\ldots,n}$ of the Stokes system in the whole space $\mathbb{R}^n$ has the same decay properties as that for the Laplace equation, namely (for $n \geq 3$)

$$|U_{ij}(x)| \lesssim |x|^{2-n}. \tag{1.3}$$

(We denote $A \lesssim B$ if there is some constant C so that $A \leq CB$.) As a result, when the force is small (of order ϵ) and sufficiently localized (i.e. the force decays sufficiently fast), one can construct the solutions to the Navier–Stokes equations with the same decay

$$|u_i(x)| \lesssim \epsilon \langle x \rangle^{2-n}, \quad \langle x \rangle := (2 + |x|^2)^{1/2}. \tag{1.4}$$

By a standard cut-off argument, one can get solutions with the same decay in an exterior domain (see [4]).