Advances in Mathematics 323 (2018) 367–410

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Ramanujan coverings of graphs

Chris Hall [a,1], Doron Puder [b,*,2], William F. Sawin [c,3]

[a] *Department of Mathematics, University of Wyoming, Laramie, WY 82071, USA*
[b] *School of Mathematics, Institute for Advanced Study, Einstein Drive, Princeton, NJ 08540, USA*
[c] *Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA*

A R T I C L E   I N F O

Article history:
Received 22 February 2016
Accepted 18 October 2017
Available online 7 November 2017
Communicated by Benjamin Sudakov

Keywords:
Expander graphs
Ramanujan graphs
Interlacing polynomials
Matching polynomial

A B S T R A C T

Let G be a finite connected graph, and let ρ be the spectral radius of its universal cover. For example, if G is k-regular then $\rho = 2\sqrt{k-1}$. We show that for every r, there is an r-covering (a.k.a. an r-lift) of G where all the new eigenvalues are bounded from above by ρ. It follows that a bipartite Ramanujan graph has a Ramanujan r-covering for every r. This generalizes the $r = 2$ case due to Marcus, Spielman and Srivastava [26].

Every r-covering of G corresponds to a labeling of the edges of G by elements of the symmetric group S_r. We generalize this notion to labeling the edges by elements of various groups and present a broader scenario where Ramanujan coverings are guaranteed to exist.

In particular, this shows the existence of richer families of bipartite Ramanujan graphs than was known before. Inspired by [26], a crucial component of our proof is the existence of interlacing families of polynomials for complex reflection groups. The core argument of this component is taken from [27].

* Corresponding author.
E-mail addresses: chall14@uwyo.edu (C. Hall), doronpuder@gmail.com (D. Puder), wsawin@math.princeton.edu (W.F. Sawin).

[1] Author Hall was partially supported by Simons Foundation award 245619 and IAS NSF grant DMS-1128155.

[2] Author Puder was supported by the Rothschild fellowship and by the National Science Foundation under agreement No. CCF-1412958.

[3] Author Sawin was supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE-1148900.

https://doi.org/10.1016/j.aim.2017.10.042
0001-8708/

Another important ingredient of our proof is a new generalization of the matching polynomial of a graph. We define the r-th matching polynomial of G to be the average matching polynomial of all r-coverings of G. We show this polynomial shares many properties with the original matching polynomial. For example, it is real rooted with all its roots inside $[-\rho, \rho]$.

Contents

1. Introduction

1.1. Ramanujan coverings

Throughout this paper, we assume that G is a finite, connected, undirected graph on n vertices and that A_G is its adjacency matrix. The eigenvalues of A_G are real and we denote them by

$$\lambda_n \leq \ldots \leq \lambda_2 \leq \lambda_1 = \mathfrak{pf}(G),$$