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Drinfeld center of enriched monoidal categories

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We define the Drinfeld center of a monoidal category enriched over a braided monoidal category, and show that every modular tensor category can be realized in a canonical way as the Drinfeld center of a self-enriched monoidal category. We also give a generalization of this result for important applications in physics.

1. Introduction

Enriched categories have been extensively studied in the past decades since they were introduced in [5]. Monoidal categories enriched over *symmetric* monoidal categories were also used implicitly or explicitly in the study of many categorical problems. For example, linear monoidal categories are enriched over the symmetric monoidal category of

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vector spaces. However, monoidal categories enriched over *braided* monoidal categories are almost vacant in the literature. It was not until recently that a definition was written down in [2,15]. This delay is partly because categories enriched over braided monoidal categories behave poorly under Cartesian product [10], which is one of the fundamental constructions in category theory.

In fact, the notion of a monoidal category enriched over a braided monoidal category is not as poor as it first looks. We show that not only one is able to generalize the Drinfeld center of a monoidal category to that of an enriched monoidal category (see Definition 4.2), but also this generalization shares many nice properties with the ordinary one, for example, it is an enriched braided monoidal category (see Theorem 4.4 and Definition 3.4).

More importantly, this notion leads to a positive answer to the following question: given a modular tensor category $\mathcal{C}$, is there any mathematical object whose "center" is $\mathcal{C}$? This question is crucial to the study of 2+1D TQFT such as Chern–Simons theory, Reshetikhin–Turaev extended TQFT [7,8,16] and topological orders with gapless edges [12]. We show that a modular tensor category (more generally, a nondegenerate braided fusion category) $\mathcal{C}$ can be realized in a canonical way as the Drinfeld center of a self-enriched monoidal category (see Corollary 4.9). We also give a generalization of this result in Corollary 5.4, which has an important application in physics [12].

2. Enriched (monoidal) categories

First, we recall the notion of a (monoidal) category enriched over a (braided) monoidal category. See [11,15] and references therein. Let $\mathcal{B}$ be a monoidal category with tensor unit $\mathbf{1}$ and tensor product $\otimes : \mathcal{B}\times\mathcal{B} \to \mathcal{B}$. We denote the identity morphism by $1 : x \to x$ for all $x \in \mathcal{B}$. The notation id_x is reserved for something else.

A *category* $\mathcal{C}^\sharp$ *enriched over* $\mathcal{B}$ consists of a set of objects $Ob(\mathcal{C}^\sharp)$, a hom object $\mathcal{C}^\sharp(x,y) \in \mathcal{B}$ for every pair $x,y \in \mathcal{C}^\sharp$, a morphism (the identity morphism) $\mathrm{id}_x : \mathbf{1} \to \mathcal{C}^\sharp(x,x)$ for every $x \in \mathcal{C}^\sharp$ and a morphism (the composition law) $\circ : \mathcal{C}^\sharp(y,z) \otimes \mathcal{C}^\sharp(x,y) \to \mathcal{C}^\sharp(x,z)$ for every triple $x,y,z \in \mathcal{C}^\sharp$ rendering the following diagrams commutative for $x,y,z,w \in \mathcal{C}^\sharp$:

$$\begin{array}{ccccc} & & \mathcal{C}^\sharp(x,y)\otimes\mathcal{C}^\sharp(x,x) & & \\ & \nearrow^{1\otimes\mathrm{id}_x} & & \searrow^{\circ} & \\ \mathcal{C}^\sharp(x,y) & & \xrightarrow{\quad\quad 1 \quad\quad} & & \mathcal{C}^\sharp(x,y), \end{array}$$

$$\begin{array}{ccccc} & & \mathcal{C}^\sharp(y,y)\otimes\mathcal{C}^\sharp(x,y) & & \\ & \nearrow^{\mathrm{id}_y\otimes 1} & & \searrow^{\circ} & \\ \mathcal{C}^\sharp(x,y) & & \xrightarrow{\quad\quad 1 \quad\quad} & & \mathcal{C}^\sharp(x,y), \end{array}$$