Advances in Mathematics 323 (2018) 427–485

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

# Spectral analysis of random-to-random Markov chains ☆

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A R T I C L E  I N F O

*Article history:*
Received 1 December 2016
Received in revised form 1 October 2017
Accepted 25 October 2017
Available online xxxx
Communicated by Ezra Miller

*Keywords:*
Random-to-random shuffling
Spectrum
Discrete Markov chain
Representation theory
Symmetric group

A B S T R A C T

We compute the eigenvalues and eigenspaces of random-to-random Markov chains. We use a family of maps which reveal a remarkable recursive structure of the eigenspaces, yielding an explicit and effective construction of all eigenbases starting from bases of the kernels.



**Contents**



☆ The second author acknowledges the support of a Natural Sciences and Engineering Research Council of Canada (NSERC) Discovery Grant.

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https://doi.org/10.1016/j.aim.2017.10.034

## 1. Introduction

Random-to-random Markov chains, also known as *random insertion* [34] and *random-to-random insertion* [36], describe the random evolution of $n$ (ordered) objects, some of which may be identical, if someone repeatedly removes an object at random and puts it back at a random position. One can think of the objects as being books on a shelf, entries in a database, characters in a word, or cards in a deck of cards. As such, the random-to-random Markov chain constitutes a canonical card shuffling model, and can be thought of as sequentially applying a random-to-top shuffle (choose a card at random and move it to the top) and a top-to-random shuffle (move the top card to a random position) at each step.

There are several fundamental questions concerning random-to-random shuffles that have withstood analysis. This is quite striking considering that these same questions have been answered for the related random-to-top and top-to-random shuffles, which play an instrumental role in the theory of card shuffling [11] and the theory of random walks on groups and semigroups [3,5,14,23,24,31]. This paper provides answers to some of these questions and new tools to investigate others.

*Our results.* This paper studies random-to-random Markov chains through a spectral lens. We introduce a family of maps that allow us to compute the eigenvalues and eigenspaces of random-to-random Markov chains. Our maps reveal a remarkable recursive structure of the eigenspaces, and yield an explicit and effective construction of all eigenbases starting from bases of the kernels.

Fig. 1 gives a high-level description of how our maps exploit the recursive nature of random-to-random Markov chains to construct its eigenbases. This paper introduces