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An embedding theorem for tangent categories ☆

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ABSTRACT

Tangent categories were introduced by Rosický as a categorical setting for differential structures in algebra and geometry; in recent work of Cockett, Crutwell and others, they have also been applied to the study of differential structure in computer science. In this paper, we prove that every tangent category admits an embedding into a representable tangent category—one whose tangent structure is given by exponentiating by a free-standing tangent vector, as in, for example, any well-adapted model of Kock and Lawvere's synthetic differential geometry. The key step in our proof uses a coherence theorem for tangent categories due to Leung to exhibit tangent categories as a certain kind of enriched category.

1. Introduction

Tangent categories, introduced by Rosický in [23], provide an category-theoretic setting for differential structures in geometry, algebra and computer science. A tangent structure on a category $\mathcal{C}$ comprises a functor $T\colon \mathcal{C} \to \mathcal{C}$ together with associated natural transformations—for example a natural transformation $p\colon T \Rightarrow 1$ making each TM

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into an object over M—which capture just those properties of the "tangent bundle" functor on the category **Man** of smooth manifolds that are necessary to develop a reasonable abstract differential calculus. The canonical example is **Man** itself, but others include the category of schemes (using the Zariski tangent spaces), the category of convenient manifolds [2] and, in computer science, any model of Ehrhard and Regnier's *differential λ-calculus* [10].

A more powerful category-theoretic approach to differential structures is the *synthetic differential geometry* developed by Kock, Lawvere, Dubuc and others [9,18,22]. It is more powerful because it presupposes more: a so-called "well-adapted" model of synthetic differential geometry is a Grothendieck topos $\mathcal{E}$ equipped with a full embedding $\iota \colon \mathbf{Man} \to \mathcal{E}$ of the category of smooth manifolds, obeying axioms which, among other things, assert that the affine line $R = \iota(\mathbb{R})$ has enough nilpotent elements to detect the differential structure. In particular, in a well-adapted model, the tangent bundle of a smooth manifold M is determined by the cartesian closed structure of $\mathcal{E}$ through the equation $\iota(TM) = \iota(M)^D$; here, D is the "disembodied tangent vector", which in the internal logic of $\mathcal{E}$ comprises the elements of R which square to zero.

Any model $\mathcal{E}$ of synthetic differential geometry gives rise to a tangent category, whose underlying category comprises the *microlinear* objects [22, Chapter V] of $\mathcal{E}$ (among which are found the embeddings $\iota(M)$ of manifolds) and whose "tangent bundle" functor is $(-)^D$; see [6, Section 5]. This raises the question of whether any tangent category can be embedded into the microlinear objects of a well-adapted model of synthetic differential geometry; and while this is probably too much to ask, it has been conjectured that any tangent category should at least be embeddable in a *representable* tangent category—one whose "tangent bundle" functor is of the form $(-)^D$. The goal of this article is to prove this conjecture.

Our approach uses ideas of enriched category theory [17]. By exploiting Leung's coherence result [20] for tangent categories, we are able to describe a cartesian closed category $\mathcal{E}$ such that tangent categories are the same thing as $\mathcal{E}$-enriched categories admitting certain *powers* [17, Section 3.7]—a kind of enriched-categorical limit. Standard enriched category theory then shows that, for any small $\mathcal{E}$-category $\mathcal{C}$, the $\mathcal{E}$-category of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ is complete, cocomplete and cartesian closed as a $\mathcal{E}$-category. Completeness means that, in particular, $[\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ bears the powers necessary for tangent structure; but cocompleteness and cartesian closure allow these powers to be computed as internal homs $(-)^D$, so that any presheaf $\mathcal{E}$-category bears *representable* tangent structure. It follows that, for any (small) tangent category $\mathcal{C}$, the $\mathcal{E}$-categorical Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{E}]$ is a full embedding of $\mathcal{C}$ into a representable tangent category.

Beyond allowing an outstanding conjecture to be settled, we believe that the enriched-categorical approach to tangent structure has independent value, which will be explored further in future work. In one direction, the category $\mathcal{E}$ over which our enrichment exists admits an abstract version of the *Campbell–Baker–Hausdorff* construction by which a Lie algebra can be formally integrated to a formal group law (i.e., encoding the purely algebraic part of Lie's theorems). Via enrichment, this construction can be transported