

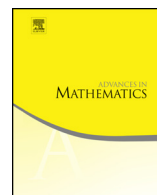


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# Two-parameter version of Bourgain's inequality: Rational frequencies <sup>☆</sup>

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## ABSTRACT

Our aim is to establish the first two-parameter version of Bourgain's maximal logarithmic inequality on  $L^2(\mathbb{R}^2)$  for the rational frequencies. We achieve this by introducing a variant of a two-parameter Rademacher–Menschov inequality. The method allows us to control an oscillation seminorm as well.

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## 1. Introduction

Let  $A_n = (-2^{-n-1}, 2^{-n-1})$  for  $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . Suppose that  $\Lambda \subset \mathbb{R}$  is a finite set satisfying the following separation condition: for any  $\lambda, \lambda' \in \Lambda$ , if  $\lambda \neq \lambda'$  then

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$$|\lambda - \lambda'| \geq 1. \tag{1}$$

In [4], Bourgain established the following lemma.

**Logarithmic lemma.** *There exists a constant  $C > 0$  such that for each  $f \in L^2(\mathbb{R})$  we have*

$$\left\| \sup_{n \in \mathbb{N}_0} \left| \sum_{\lambda \in \Lambda} \mathcal{F}^{-1}(\mathbb{1}_{A_n^\lambda} \mathcal{F}f) \right| \right\|_{L^2} \leq C (\log |\Lambda|)^2 \|f\|_{L^2} \tag{2}$$

where  $A_n^\lambda = \lambda + A_n$ , and  $\mathcal{F}$  is the Fourier transform operator on  $\mathbb{R}$ . Moreover, the implied constant is independent of the cardinality of the set  $\Lambda$ .

This logarithmic lemma was introduced by Bourgain to reduce some problems in ergodic theory having a number theoretic nature to questions in harmonic analysis (compare [2,3] with [4]). To be more precise, let  $(X, \mathcal{B}, \mu)$  be a  $\sigma$ -finite measure space and let  $T : X \rightarrow X$  be an invertible measure preserving transformation. The classical Birkhoff’s theorem (see [1]) states that for any  $f \in L^p(X, \mu)$  with  $p \geq 1$  the averages

$$A_N f(x) := \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x)$$

converges  $\mu$ -almost everywhere. With the aid of the logarithmic lemma Bourgain proved the pointwise convergence of

$$A_N^{\mathcal{P}} f(x) := \frac{1}{N} \sum_{n=0}^{N-1} f(T^{\mathcal{P}(n)} x)$$

for all  $f \in L^p(X, \mu)$  and  $p > 1$ ; where,  $\mathcal{P}$  is a polynomial with integer coefficients. The lemma was applied to the sets

$$\mathcal{R}_s = \{a/q \in [0, 1] \cap \mathbb{Q} : (a, q) = 1, \text{ and } 2^s \leq q < 2^{s+1}\}$$

giving an acceptable loss with respect to  $s$  in (2) of the order  $s^2$  since  $|\mathcal{R}_s| \leq 4^s$  (see [4] for more details).

In fact, in [4] the logarithmic lemma was proven in a much stronger form: for general frequencies without the separation condition (1). Not long afterwards, it was observed by Lacey (see [14]) that if  $\Lambda \subset Q^{-1}\mathbb{Z}$  for some  $Q \in \mathbb{N}$  and satisfies separation condition, then

$$\left\| \sup_{n \in \mathbb{N}_0} \left| \sum_{\lambda \in \Lambda} \mathcal{F}^{-1}(\mathbb{1}_{A_n^\lambda} \mathcal{F}f) \right| \right\|_{L^2} \leq C \log \log (Q\sqrt{|\Lambda|}) \|f\|_{L^2}. \tag{3}$$

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