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Two-parameter version of Bourgain's inequality: Rational frequencies ☆

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ABSTRACT

Our aim is to establish the first two-parameter version of Bourgain's maximal logarithmic inequality on $L^2(\mathbb{R}^2)$ for the rational frequencies. We achieve this by introducing a variant of a two-parameter Rademacher–Menschov inequality. The method allows us to control an oscillation seminorm as well.

1. Introduction

Let $A_n = (-2^{-n-1}, 2^{-n-1})$ for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Suppose that $\Lambda \subset \mathbb{R}$ is a finite set satisfying the following separation condition: for any $\lambda, \lambda' \in \Lambda$, if $\lambda \neq \lambda'$ then

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$$|\lambda - \lambda'| \geq 1. \tag{1}$$

In [4], Bourgain established the following lemma.

Logarithmic lemma. *There exists a constant $C > 0$ such that for each $f \in L^2(\mathbb{R})$ we have*

$$\Big\| \sup_{n\in\mathbb{N}_0} \Big| \sum_{\lambda\in\Lambda} \mathcal{F}^{-1}\big(\mathbb{1}_{A_n^\lambda}\mathcal{F}f\big)\Big|\Big\|_{L^2} \leq C\big(\log|\Lambda|\big)^2\|f\|_{L^2} \tag{2}$$

where $A_n^\lambda = \lambda + A_n$, and $\mathcal{F}$ is the Fourier transform operator on $\mathbb{R}$. Moreover, the implied constant is independent of the cardinality of the set Λ.

This logarithmic lemma was introduced by Bourgain to reduce some problems in ergodic theory having a number theoretic nature to questions in harmonic analysis (compare [2,3] with [4]). To be more precise, let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let $T : X \to X$ be an invertible measure preserving transformation. The classical Birkhoff's theorem (see [1]) states that for any $f \in L^p(X, \mu)$ with $p \geq 1$ the averages

$$A_N f(x) := \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)$$

converges μ-almost everywhere. With the aid of the logarithmic lemma Bourgain proved the pointwise convergence of

$$A_N^{\mathcal{P}} f(x) := \frac{1}{N}\sum_{n=0}^{N-1} f\big(T^{\mathcal{P}(n)}x\big)$$

for all $f \in L^p(X, \mu)$ and $p > 1$; where, $\mathcal{P}$ is a polynomial with integer coefficients. The lemma was applied to the sets

$$\mathscr{R}_s = \big\{a/q \in [0,1] \cap \mathbb{Q} : (a,q) = 1, \text{ and } 2^s \leq q < 2^{s+1}\big\}$$

giving an acceptable loss with respect to s in (2) of the order s^2 since $|\mathscr{R}_s| \leq 4^s$ (see [4] for more details).

In fact, in [4] the logarithmic lemma was proven in a much stronger form: for general frequencies without the separation condition (1). Not long afterwards, it was observed by Lacey (see [14]) that if $\Lambda \subset Q^{-1}\mathbb{Z}$ for some $Q \in \mathbb{N}$ and satisfies separation condition, then

$$\Big\| \sup_{n\in\mathbb{N}_0} \Big| \sum_{\lambda\in\Lambda} \mathcal{F}^{-1}\big(\mathbb{1}_{A_n^\lambda}\mathcal{F}f\big)\Big|\Big\|_{L^2} \leq C\log\log\big(Q\sqrt{|\Lambda|}\big)\|f\|_{L^2}. \tag{3}$$