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The moduli space of polynomial maps and their fixed-point multipliers ☆

Toshi Sugiyama

Nada High School, Uozaki-kita-machi 8-5-1, Higashinada, Kobe 658-0082, Japan

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ABSTRACT

We consider the family MP_d of affine conjugacy classes of polynomial maps of one complex variable with degree $d \geq 2$, and study the map $\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d \subset \mathbb{C}^d/\mathfrak{S}_d$ which maps each $f \in \mathrm{MP}_d$ to the set of fixed-point multipliers of f. We show that the local fiber structure of the map Φ_d around $\bar{\lambda} \in \widetilde{\Lambda}_d$ is completely determined by certain two sets $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$ which are subsets of the power set of $\{1, 2, \ldots, d\}$. Moreover for any $\bar{\lambda} \in \widetilde{\Lambda}_d$, we give an algorithm for counting the number of elements of each fiber $\Phi_d^{-1}(\bar{\lambda})$ only by using $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$. It can be carried out in finitely many steps, and often by hand.

1. Introduction

Let MP_d be the family of affine conjugacy classes of polynomial maps of one complex variable with degree $d \geq 2$, and $\mathbb{C}^d/\mathfrak{S}_d$ the set of unordered collections of d complex

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E-mail address: sugiyama.toshi@gmail.com.

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numbers. Then the aim of this paper is to give a *complete description* of the fiber structure of the map

$$\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d \subset \mathbb{C}^d/\mathfrak{S}_d$$

which maps each $f \in \mathrm{MP}_d$ to the set of fixed-point multipliers of f, except where $f \in \mathrm{MP}_d$ has multiple fixed points.

Since multipliers of fixed points have played a central role in the study of the complex dynamics, it is natural to ask to what extent fixed-point multipliers of f determine the original map f. For polynomial maps, since the set of fixed-point multipliers is invariant under the action of affine transformations, the question is to count the number of affine conjugacy classes of polynomial maps when the set of its fixed-point multipliers are given. It is formulated in the following form: how many elements there are on each fiber of the above map $\Phi_d : \mathrm{MP}_d \to \mathbb{C}^d/\mathfrak{S}_d$. Here, since the set of fixed-point multipliers always satisfies a certain relation by the fixed point theorem (see Proposition 1.1), the image of Φ_d is contained in a certain hyperplane $\widetilde{\Lambda}_d$ in $\mathbb{C}^d/\mathfrak{S}_d$. Hence the main object of our study is the map $\Phi_d : \mathrm{MP}_d \to \widetilde{\Lambda}_d$.

For $d = 2$, it is easily verified that Φ_2 is bijective. In the case $d = 3$, Milnor [11] showed that Φ_3 is also bijective, which was the starting point of his study of the complex dynamics of cubic polynomials. For $d \geq 4$, Fujimura and Nishizawa have long studied the map Φ_d in their series of papers such as [16], [3] and [4]. Especially their achievement is summarized in Fujimura's paper [4], which includes the following:

- Φ_d is not surjective for $d \geq 4$. Moreover for $d = 4$ or 5, she found all $\bar{\lambda} \in \widetilde{\Lambda}_d$ whose inverse image of Φ_d is empty.
- Generic fiber of Φ_d consists of $(d-2)!$ points. Moreover if $\Phi_d^{-1}(\bar{\lambda})$ is finite, then $\#\left(\Phi_d^{-1}(\bar{\lambda})\right) \leq (d-2)!$ always holds.
- For $d = 4$, she found $\#\left(\Phi_4^{-1}(\bar{\lambda})\right)$ for all $\bar{\lambda} \in \widetilde{\Lambda}_4$.

Here, we denote the cardinality of a set X by $\#(X)$. Similar results for rational maps are given by Milnor in [13, p. 152, Problem 12-d] and [12].

Based on the results above, this paper provides an algorithm for counting the number of elements of each fiber $\Phi_d^{-1}(\bar{\lambda})$ for *all* $\bar{\lambda} = \{\lambda_1, \ldots, \lambda_d\} \in \widetilde{\Lambda}_d$ and for *all* $d \geq 4$ except when $\lambda_i = 1$ for some i. In practice, for each $\lambda = (\lambda_1, \ldots, \lambda_d) \in \Lambda_d \subset \mathbb{C}^d$ with $\lambda_i \neq 1$, certain two subsets $\mathcal{I}(\lambda), \mathcal{K}(\lambda)$ of the power set of $\{1, 2, \ldots, d\}$ are defined, and the number of elements of a fiber $\Phi_d^{-1}(\bar{\lambda})$ is completely determined by $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$. Moreover we give an algorithm for counting the number $\#\left(\Phi_d^{-1}\left(\bar{\lambda}\right)\right)$ only by using $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$ (see Main Theorems I, III, Definition 1.7 and Section 2). The algorithm can be carried out in finitely many steps, and only by hand. Moreover in Main Theorem II we show that the local fiber structure of Φ_d around $\bar{\lambda}$ is also determined by $\mathcal{I}(\lambda)$ and $\mathcal{K}(\lambda)$.

We shall provide some more concerning results.