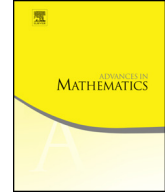




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# The moduli space of polynomial maps and their fixed-point multipliers<sup>☆</sup>



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## ABSTRACT

We consider the family  $MP_d$  of affine conjugacy classes of polynomial maps of one complex variable with degree  $d \geq 2$ , and study the map  $\Phi_d : MP_d \rightarrow \tilde{\Lambda}_d \subset \mathbb{C}^d/\mathfrak{S}_d$  which maps each  $f \in MP_d$  to the set of fixed-point multipliers of  $f$ . We show that the local fiber structure of the map  $\Phi_d$  around  $\bar{\lambda} \in \tilde{\Lambda}_d$  is completely determined by certain two sets  $\mathcal{I}(\lambda)$  and  $\mathcal{K}(\lambda)$  which are subsets of the power set of  $\{1, 2, \dots, d\}$ . Moreover for any  $\bar{\lambda} \in \tilde{\Lambda}_d$ , we give an algorithm for counting the number of elements of each fiber  $\Phi_d^{-1}(\bar{\lambda})$  only by using  $\mathcal{I}(\lambda)$  and  $\mathcal{K}(\lambda)$ . It can be carried out in finitely many steps, and often by hand.

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## 1. Introduction

Let  $MP_d$  be the family of affine conjugacy classes of polynomial maps of one complex variable with degree  $d \geq 2$ , and  $\mathbb{C}^d/\mathfrak{S}_d$  the set of unordered collections of  $d$  complex

<sup>☆</sup> This work was done when the author was a graduate student of Kyoto University.

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numbers. Then the aim of this paper is to give a *complete description* of the fiber structure of the map

$$\Phi_d : \text{MP}_d \rightarrow \tilde{\Lambda}_d \subset \mathbb{C}^d / \mathfrak{S}_d$$

which maps each  $f \in \text{MP}_d$  to the set of fixed-point multipliers of  $f$ , except where  $f \in \text{MP}_d$  has multiple fixed points.

Since multipliers of fixed points have played a central role in the study of the complex dynamics, it is natural to ask to what extent fixed-point multipliers of  $f$  determine the original map  $f$ . For polynomial maps, since the set of fixed-point multipliers is invariant under the action of affine transformations, the question is to count the number of affine conjugacy classes of polynomial maps when the set of its fixed-point multipliers are given. It is formulated in the following form: how many elements there are on each fiber of the above map  $\Phi_d : \text{MP}_d \rightarrow \mathbb{C}^d / \mathfrak{S}_d$ . Here, since the set of fixed-point multipliers always satisfies a certain relation by the fixed point theorem (see [Proposition 1.1](#)), the image of  $\Phi_d$  is contained in a certain hyperplane  $\tilde{\Lambda}_d$  in  $\mathbb{C}^d / \mathfrak{S}_d$ . Hence the main object of our study is the map  $\Phi_d : \text{MP}_d \rightarrow \tilde{\Lambda}_d$ .

For  $d = 2$ , it is easily verified that  $\Phi_2$  is bijective. In the case  $d = 3$ , Milnor [\[11\]](#) showed that  $\Phi_3$  is also bijective, which was the starting point of his study of the complex dynamics of cubic polynomials. For  $d \geq 4$ , Fujimura and Nishizawa have long studied the map  $\Phi_d$  in their series of papers such as [\[16\]](#), [\[3\]](#) and [\[4\]](#). Especially their achievement is summarized in Fujimura’s paper [\[4\]](#), which includes the following:

- $\Phi_d$  is not surjective for  $d \geq 4$ . Moreover for  $d = 4$  or  $5$ , she found all  $\bar{\lambda} \in \tilde{\Lambda}_d$  whose inverse image of  $\Phi_d$  is empty.
- Generic fiber of  $\Phi_d$  consists of  $(d - 2)!$  points. Moreover if  $\Phi_d^{-1}(\bar{\lambda})$  is finite, then  $\#(\Phi_d^{-1}(\bar{\lambda})) \leq (d - 2)!$  always holds.
- For  $d = 4$ , she found  $\#(\Phi_4^{-1}(\bar{\lambda}))$  for all  $\bar{\lambda} \in \tilde{\Lambda}_4$ .

Here, we denote the cardinality of a set  $X$  by  $\#(X)$ . Similar results for rational maps are given by Milnor in [\[13, p. 152, Problem 12-d\]](#) and [\[12\]](#).

Based on the results above, this paper provides an algorithm for counting the number of elements of each fiber  $\Phi_d^{-1}(\bar{\lambda})$  for all  $\bar{\lambda} = \{\lambda_1, \dots, \lambda_d\} \in \tilde{\Lambda}_d$  and for all  $d \geq 4$  except when  $\lambda_i = 1$  for some  $i$ . In practice, for each  $\lambda = (\lambda_1, \dots, \lambda_d) \in \Lambda_d \subset \mathbb{C}^d$  with  $\lambda_i \neq 1$ , certain two subsets  $\mathcal{I}(\lambda), \mathcal{K}(\lambda)$  of the power set of  $\{1, 2, \dots, d\}$  are defined, and the number of elements of a fiber  $\Phi_d^{-1}(\bar{\lambda})$  is completely determined by  $\mathcal{I}(\lambda)$  and  $\mathcal{K}(\lambda)$ . Moreover we give an algorithm for counting the number  $\#(\Phi_d^{-1}(\bar{\lambda}))$  only by using  $\mathcal{I}(\lambda)$  and  $\mathcal{K}(\lambda)$  (see [Main Theorems I, III, Definition 1.7](#) and [Section 2](#)). The algorithm can be carried out in finitely many steps, and only by hand. Moreover in [Main Theorem II](#) we show that the local fiber structure of  $\Phi_d$  around  $\bar{\lambda}$  is also determined by  $\mathcal{I}(\lambda)$  and  $\mathcal{K}(\lambda)$ .

We shall provide some more concerning results.

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