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# Multiplicity-free Kronecker products of characters of the symmetric groups

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We provide a classification of multiplicity-free inner tensor products of irreducible characters of symmetric groups, thus confirming a conjecture of Bessenrodt. Concurrently, we classify all multiplicity-free inner tensor products of skew characters of the symmetric groups. We also provide formulae for calculating the decomposition of these tensor products.



## 1. Introduction

The inner and outer tensor products of irreducible characters of the symmetric groups (or equivalently of Schur functions) have been of central interest in representation theory and algebraic combinatorics since the landmark papers of Littlewood and Richardson [19] and Murnaghan [23]. More recently, these coefficients have provided the centrepiece of

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geometric complexity theory in an approach that seeks to settle the $\mathsf{P}$ versus $\mathsf{NP}$ problem [22]; it was recently shown to require not only positivity but precise information on the coefficients [6]. The Kronecker coefficients have also been found to have deep connections with quantum information theory [7].

The coefficients arising in the outer tensor product are the most well-understood. The Littlewood–Richardson rule provides an efficient positive combinatorial description for their computation. Using this algorithm, a classification of multiplicity-free outer tensor products was obtained by Stembridge [30]. This was extended to a classification of multiplicity-free skew characters by Gutschwager [15], a result equivalent to the classification of multiplicity-free products of Schubert classes obtained around the same time by Thomas and Yong [31].

By contrast, the coefficients arising in the inner tensor product are much less well-understood; indeed, they have been described as 'perhaps the most challenging, deep and mysterious objects in algebraic combinatorics' [24]. The determination of these coefficients has been described by Richard Stanley as 'one of the main problems in the combinatorial representation theory of the symmetric group' [29]. While 'no satisfactory answer to this question is known' [18] there have, over many decades, been a number of contributions made towards computing special products (such as those labelled by 2-line or hook partitions [4,5,27,25,26] or certain powers [12]) or the multiplicity of special constituents (for example those with few homogeneous components [2,3]).

In 1999, Bessenrodt conjectured a classification of multiplicity-free Kronecker products of irreducible characters of the symmetric groups. Mainly using results of Remmel, Saxl and Vallejo, it was shown at that time that the products on the conjectured list were indeed multiplicity-free and the conjecture was verified by computer calculations for all $n \leqslant 40$. Since then, multiplicity-free Kronecker products have been studied in [1,5,14,20]. In this paper we prove that the classification list is indeed complete for all $n \in \mathbb{N}$ and hence confirm the conjecture, that is, we have the following result:

**Theorem 1.1.** *Let $\lambda, \mu$ be partitions of $n \in \mathbb{N}$. Then the Kronecker product $[\lambda] \cdot [\mu]$ of the irreducible characters $[\lambda], [\mu]$ of $\mathfrak{S}_n$ is multiplicity-free if and only if the partitions $\lambda, \mu$ satisfy one of the following conditions (up to conjugation of one or both of the partitions):*

*(1) One of the partitions is $(n)$, and the other one is arbitrary;*
*(2) one of the partitions is $(n-1,1)$, and the other one is a fat hook (here, a fat hook is a partition with at most two different parts, i.e. it is of the form $(a^b, c^d)$, $a \geqslant c$);*
*(3) $n = 2k+1$ and $\lambda = (k+1,k) = \mu$, or $n = 2k$ and $\lambda = (k,k) = \mu$;*
*(4) $n = 2k$, one of the partitions is $(k,k)$, and the other one is one of $(k+1,k-1)$, $(n-3,3)$ or a hook;*
*(5) one of the partitions is a rectangle, and the other one is one of $(n-2,2)$, $(n-2,1^2)$;*
*(6) the partition pair is one of the pairs $((3^3),(6,3))$, $((3^3),(5,4))$, and $((4^3),(6^2))$.*