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Moduli spaces of Higgs bundles on degenerating Riemann surfaces

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We prove a gluing theorem for solutions (A_0, Φ_0) of Hitchin's self-duality equations with logarithmic singularities on a rank-2 vector bundle over a noded Riemann surface Σ_0 representing a boundary point of Teichmüller moduli space. We show that every nearby smooth Riemann surface Σ_1 carries a smooth solution (A_1, Φ_1) of the self-duality equations, which may be viewed as a desingularization of (A_0, Φ_0).

1. Introduction

The moduli space of solutions to Hitchin's self-duality equations on a compact Riemann surface, by its definition primarily an object of geometric analysis, is intimately related to a number of diverse fields such as algebraic geometry, geometric topology and the emerging subject of higher Teichmüller theory. From an analytic point of view, it

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is the space of gauge equivalence classes of solutions to the system of first-order partial differential equations

$$\begin{aligned} F_A^\perp + [\Phi \wedge \Phi^*] &= 0, \\ \bar{\partial}_A \Phi &= 0 \end{aligned} \tag{1}$$

for a pair (A, Φ), where A is a unitary connection on a hermitian vector bundle E over a Riemann surface (Σ, J), and Φ is an $\mathrm{End}(E)$-valued $(1,0)$-form, the so-called Higgs field. Here $F_A^\perp$ is the trace-free part of the curvature of A, a 2-form with values in the skew-hermitian endomorphisms of E, and the adjoint Higgs field Φ^* is computed with respect to the hermitian metric on E. When restricted to a suitable slice of the action by unitary gauge transformations, Eq. (1) form a system of elliptic partial differential equations. We always assume that the genus g of the closed surface Σ is at least 2.

The moduli space $\mathcal{M}$ of solutions to the self-duality equations, first introduced by Hitchin [12] as a two-dimensional reduction of the standard self-dual Yang–Mills equations in four dimensions, shows a rich geometric structure in very different ways: as a quasi-projective variety [12,22,20], as the phase space of a completely integrable system [13,14], and (in case where the rank and degree of E are coprime) as a noncompact smooth manifold carrying a complete hyperkähler metric g_{WP} of Weil–Petersson type. Concerning the second and the last point, we mention in particular the recent work of Gaiotto, Moore and Neitzke [8,9] concerning hyperkähler metrics on holomorphic integrable systems. They describe a natural but incomplete hyperkähler metric g_0 on $\mathcal{M}$ as a leading term (the semiflat metric in the language of [7]) plus an asymptotic series of non-perturbative corrections, which decay exponentially in the distance from some fixed point in moduli space. The coefficients of these correction terms are given there in terms of a priori divergent expressions coming from a wall-crossing formalism. The thus completed hyperkähler metric is conjectured to coincide with the above mentioned metric g_{WP} on moduli space. A further motivation to study this moduli space is Sen's conjecture about the L^2-cohomology of the monopole moduli spaces [21] and the variant of it due to Hausel concerning the L^2-cohomology of $\mathcal{M}$.

The first study of the self-duality equations on higher dimensional Kähler manifolds is due to Simpson [22–24]. As shown by Donaldson [6] for Riemann surfaces (and extended to the higher dimensional case by Corlette [4]) the moduli space $\mathcal{M}$ corresponds closely to the variety of representations of (a central extension of) the fundamental group of Σ into the Lie group $\mathrm{SL}(r, \mathbb{C})$, $r = \mathrm{rk}(E)$ (see [10] and references therein), and thus permits to be studied by more algebraic and topological methods.

In recent joint work, Mazzeo, Weiß, Witt, and the present author started an investigation of the large scale structure of this moduli space, resulting so far in a precise description of the profile of solutions in the limit of "large" Higgs fields, i.e. when $\|\Phi\|_{L^2} \to \infty$, cf. [18,17]. Moreover, a geometric compactification of $\mathcal{M}$ was obtained, which consists in adding to $\mathcal{M}$ configurations (A, Φ), that are singular in a finite set of points and admit an interpretation as so-called parabolic Higgs bundles.