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Functional distribution monads in functional-analytic contexts

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ABSTRACT

We give a general categorical construction that yields several monads of measures and distributions as special cases, alongside several monads of filters. The construction takes place within a categorical setting for generalized functional analysis, called a *functional-analytic context*, formulated in terms of a given monad or algebraic theory $\mathscr{T}$ enriched in a closed category $\mathscr{V}$. By employing the notion of *commutant* for enriched algebraic theories and monads, we define the *functional distribution monad* associated to a given functional-analytic context. We establish certain general classes of examples of functional-analytic contexts in cartesian closed categories $\mathscr{V}$, wherein $\mathscr{T}$ is the theory of R-modules or R-affine spaces for a given ring or rig R in $\mathscr{V}$, or the theory of *R-convex spaces* for a given preordered ring R in $\mathscr{V}$. We prove theorems characterizing the functional distribution monads in these contexts, and on this basis we establish several specific examples of functional distribution monads.

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1. Introduction

Through work of Lawvere [32], Świrszcz [53], Giry [17] and many others it has become clear that various kinds of measures and distributions give rise to monads; see, for example, [20,8,37,31,38,1]. The earliest work in this regard considered monads $\mathbb{M}$ for which each free $\mathbb{M}$-algebra MV is a space of probability measures on a space V, be it a measurable space [32,17] or a compact Hausdorff space [53], for example. In the literature, one does not find a monad capturing *arbitrary* measures on a given class of spaces, whereas one can capture measures of *compact support* [38, 7.1.7] or *bounded support* [37], as well as Schwartz distributions of compact support [38, 7.1.6] [50] [48, II.3.6].

In the present paper, we give a general categorical construction which yields several (new and old) monads of measures and distributions as special cases. The general construction takes place within an abstract axiomatic setting for generalized functional analysis, called a *functional-analytic context* (4.1, 4.2), and in such a context we define an associated monad that we call the *functional distribution monad* (4.5). By considering various particular contexts, we obtain several particular monads of measures or distributions, as well as certain monads of *filters*,[2] as instances of an abstract construction with a functional-analytic flavour.

Our starting point is the *Riesz–Schwartz dualization paradigm*, which has perhaps its purest expression in *cartesian closed* categories. Given a commutative ring object R in a cartesian closed category $\mathscr{V}$ with finite limits, one can form for each object V of $\mathscr{V}$ a canonical 'function space', namely the internal hom $[V, R]$, and we define

$$DV = R\text{-Mod}([V, R], R) \tag{1.0.i}$$

to be the subobject of $[[V, R], R]$ described by the equations that characterize R-linear morphisms $\mu : [V, R] \to R$. This construction was employed in the context of synthetic differential geometry [50,28,48], and it yields a monad $\mathbb{D}$ on $\mathscr{V}$. When $\mathscr{V}$ and R are suitably chosen, the space of all compactly supported Radon measures on a locally compact Hausdorff space V is recovered as an example of one of the free $\mathbb{D}$-algebras DV [38, 7.1.6], and one can similarly capture compactly supported Schwartz distributions on a smooth manifold V; see [38, 7.1.6] and [48, II.3.6]. But in these examples, the space DV is by no means locally compact (nor, respectively, a smooth manifold), and so one must embed the categories of locally compact spaces and smooth manifolds into larger categories (e.g. convergence spaces, Frölicher spaces, diffeological spaces, and various toposes in synthetic differential geometry; see §2).

A generalization of the formula (1.0.i) was employed by Kock [31] and by the author [38], wherein for a given commutative $\mathscr{V}$-enriched monad $\mathbb{T}$ on any suitable closed category $\mathscr{V}$ we set

[2] Formal connections between the ultrafilter monad and notions of distribution or integral are noted in [30,35].