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Estimates for measures of lower dimensional sections of convex bodies

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ABSTRACT

We present an alternative approach to some results of Koldobsky on measures of sections of symmetric convex bodies, which allows us to extend them to the not necessarily symmetric setting. We prove that if K is a convex body in $\mathbb{R}^n$ with $0 \in \text{int}(K)$ and μ is a measure on $\mathbb{R}^n$ with a locally integrable non-negative density g on $\mathbb{R}^n$, then

$$\mu(K) \leq \left(c\sqrt{n-k}\right)^k \max_{F \in G_{n,n-k}} \mu(K \cap F) \cdot |K|^{\frac{k}{n}}$$

for every $1 \leq k \leq n-1$. Also, if μ is even and log-concave, and if K is a symmetric convex body in $\mathbb{R}^n$ and D is a compact subset of $\mathbb{R}^n$ such that $\mu(K \cap F) \leq \mu(D \cap F)$ for all $F \in G_{n,n-k}$, then

$$\mu(K) \leq (ckL_{n-k})^k \mu(D),$$

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where L_s is the maximal isotropic constant of a convex body in $\mathbb{R}^s$. Our method employs a generalized Blaschke–Petkantschin formula and estimates for the dual affine quermassintegrals.

1. Introduction

In this article we discuss lower dimensional versions of the slicing problem and of the Busemann–Petty problem, both in the classical setting and in the generalized setting of arbitrary measures in place of volume, which was put forward by Koldobsky for the slicing problem and by Zvavitch for the Busemann–Petty problem. We introduce an alternative approach which is based on the generalized Blaschke–Petkantschin formula and on asymptotic estimates for the dual affine quermassintegrals.

The classical slicing problem asks if there exists an absolute constant $C_1 > 0$ such that for every $n \geq 1$ and every convex body K in $\mathbb{R}^n$ with center of mass at the origin (we call these convex bodies centered) one has

$$|K|^{\frac{n-1}{n}} \leq C_1 \max_{\theta \in S^{n-1}} |K \cap \theta^{\perp}|. \tag{1.1}$$

It is well-known that this problem is equivalent to the question if there exists an absolute constant $C_2 > 0$ such that

$$L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leq C_2 \tag{1.2}$$

for all $n \geq 1$ (see Section 2 for background information on isotropic convex bodies and log-concave probability measures). Bourgain proved in [2] that $L_n \leq c\sqrt[4]{n}\log n$, and Klartag [11] improved this bound to $L_n \leq c\sqrt[4]{n}$. A second proof of Klartag's bound appears in [12]. From the equivalence of the two questions it follows that

$$|K|^{\frac{n-1}{n}} \leq c_1 L_n \max_{\theta \in S^{n-1}} |K \cap \theta^{\perp}| \leq c_2 \sqrt[4]{n} \max_{\theta \in S^{n-1}} |K \cap \theta^{\perp}| \tag{1.3}$$

for every centered convex body K in $\mathbb{R}^n$.

The natural generalization, the lower dimensional slicing problem, is the following question: Let $1 \leq k \leq n-1$ and let $\alpha_{n,k}$ be the smallest positive constant $\alpha > 0$ with the following property: For every centered convex body K in $\mathbb{R}^n$ one has

$$|K|^{\frac{n-k}{n}} \leq \alpha^k \max_{F \in G_{n,n-k}} |K \cap F|. \tag{1.4}$$

Is it true that there exists an absolute constant $C_3 > 0$ such that $\alpha_{n,k} \leq C_3$ for all n and k?