Advances in Mathematics 306 (2017) 905–928

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Local systems on analytic germ complements

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A R T I C L E   I N F O

Article history:
Received 28 November 2015
Accepted 2 November 2016
Available online 11 November 2016
Communicated by Karen Smith

Keywords:
Local systems
Cohomology
Jump loci
Riemann–Hilbert correspondence

A B S T R A C T

We prove that the cohomology jump loci of rank one local systems on the complement in a small ball of a germ of a complex analytic set are finite unions of torsion translates of subtori. This is a generalization of the classical Monodromy Theorem stating that the eigenvalues of the monodromy on the cohomology of the Milnor fiber of a germ of a holomorphic function are roots of unity.

1. Introduction

Let $(\mathscr{X},0) \subset (\mathbb{C}^n,0)$ be the germ of a complex analytic set. Let B be a small open ball at the origin in $\mathbb{C}^n$, and $U = B \setminus \mathscr{X}$. Up to homotopy equivalence, the open set U is uniquely determined by the germ $\mathscr{X}$. We call such open set U the small ball complement of $\mathscr{X}$.

By the local conic structure of analytic sets, U has the same homotopy type as the link complement $\partial(\bar{B}) \setminus \mathcal{L}$, where the link of $(\mathscr{X},0)$ is defined as $\mathcal{L} = \partial(\bar{B}) \cap \mathscr{X}$, and $\partial(\bar{B})$ is the boundary sphere of B. We will however keep working throughout this paper with small ball complements of analytic germs rather than link complements.

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http://dx.doi.org/10.1016/j.aim.2016.11.002

Let $\mathcal{M}_{\mathrm{B}}(U)$ be the moduli space of rank 1 local systems on U. By identifying a local system with the monodromies around loops in U, one can write $\mathcal{M}_{\mathrm{B}}(U) \cong \mathrm{Hom}(H_1(U,\mathbb{Z}),\mathbb{C}^*)$. The cohomology jump loci of U are defined as usual by

$$\mathcal{V}^i_k(U) = \left\{ L \in \mathcal{M}_{\mathrm{B}}(U) | \dim H^i(U,L) \geq k \right\}.$$

Cohomology of local systems can be computed from twisted cochain complexes on the universal cover. It is known that this implies that $\mathcal{V}^i_k(U)$ are affine $\mathbb{Z}$-schemes of finite type for any topological space U that has the homotopy type of a finite CW-complex, and that $\mathcal{V}^i_k(U)$ depend only on the homotopy type of U.

The main result is the following:

Theorem 1.1. *Let U be the small ball complement of the germ of a complex analytic set. Then each irreducible component of $\mathcal{V}^i_k(U)$ is a torsion translate of subtorus of $\mathcal{M}_{\mathrm{B}}(U)$.*

Here, by a subtorus of $\mathcal{M}_{\mathrm{B}}(U)$, we mean an affine algebraic subtorus $(\mathbb{C}^*)^p$ of the identity-containing component $\mathcal{M}_{\mathrm{B}}(U)_{\mathbf{1}}$ of $\mathcal{M}_{\mathrm{B}}(U)$. In particular, all such subtori are defined over $\mathbb{Z}$. Note that $\mathcal{M}_{\mathrm{B}}(U)_{\mathbf{1}} \cong (\mathbb{C}^*)^r$, where r is the first Betti number of U.

The homotopy type of the small ball complement of a complex singularity is much more restricted than that of the link of the singularity. Indeed, by a result of Kapovich–Kollár [16], every finitely presented group can be achieved as the fundamental group of the link of an isolated complex singularity. Hence the cohomology jump loci $\mathcal{V}^1_1$ of such links can be any affine $\mathbb{Z}$-schemes of finite type, since it is known how $\mathcal{V}^1_1$ can be defined entirely from π_1 for any topological space.

In the last section of this paper, we also consider small ball complements in singular ambient spaces.

Theorem 1.1 is the local counterpart of a similar statement proven for the cohomology jump loci of smooth complex quasi-projective varieties in [6] and of compact Kähler manifolds in [28]. These results are built on a long series of partial results due to Green–Lazarsfeld, Arapura, Simpson, Dimca–Papadima, etc. For precise references see the recent survey [7]. An interesting feature of this paper is that it does not rely on any Hodge theory, formality properties, nor on previously proven cases.

An easy consequence of the structure result for cohomology jump loci is invariance under taking the inverse, which is also the dual, of a rank one local system:

Theorem 1.2. *Let X be a smooth complex quasi-projective variety, a compact Kähler manifold, or a small ball complement of the germ of a complex analytic set. Or, more generally, let X be a topological space of homotopy type of a finite CW-complex such that the irreducible components of $\mathcal{V}^i_k(X)$ are torsion translated subtori of $\mathcal{M}_{\mathrm{B}}(X)$. Then for any rank one local system L on X, non-canonically,*

$$H^i(X,L) \cong H^i(X,L^{-1}).$$