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Extremal function for capacity and estimates of QED constants in $\mathbb{R}^n$ ☆

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ABSTRACT

This paper is devoted to the study of some fundamental problems on modulus and extremal length of curve families, capacity, and n-harmonic functions in the Euclidean space $\mathbb{R}^n$. One of the main goals is to establish the existence, uniqueness, and boundary behavior of the extremal function for the conformal capacity $\operatorname{cap}(A, B; \Omega)$ of a capacitor in $\mathbb{R}^n$. This generalizes some well known results and has its own interests in geometric function theory and potential theory. It is also used as a major ingredient in this paper to establish a sharp upper bound for the quasiextremal distance (or QED) constant $M(\Omega)$ of a domain in terms of its local boundary quasiconformal reflection constant $H(\Omega)$, a bound conjectured by Shen in the plane. Along the way, several interesting results are established for modulus and extremal length. One of them is a decomposition theorem for the extremal length $\lambda(A, B; \Omega)$ of the curve family joining two disjoint continua A and B in a domain Ω.

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Contents

1. Introduction

In this paper, we study some fundamental problems on modulus, extremal length, capacity, and n-harmonic functions in the Euclidean space. These concepts play crucial roles in geometric function theory and potential theory.

1.1. Modulus, extremal length, and capacity

Throughout this paper, we let $\mathbb{R}^n$ denote the Euclidean n-space and $\overline{\mathbb{R}}^n$ its one point compactification $\mathbb{R}^n \cup \{\infty\}$. A ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$ will be denoted by $B(x, r)$. The boundary and closure of a set A in $\overline{\mathbb{R}}^n$ are denoted by ∂A and $\overline{A}$, respectively.

For a curve family Γ in $\mathbb{R}^n$, its *(conformal) modulus* $\mathrm{mod}(\Gamma)$ is defined as

$$\mathrm{mod}(\Gamma) = \inf_{\rho \in adm(\Gamma)} \int_{\mathbb{R}^n} \rho^n dm$$

where the infimum is taken over the set, denoted by $adm(\Gamma)$, of all non-negative Borel measurable functions $\rho : \mathbb{R}^n \to \mathbb{R}$ such that $\int_\gamma \rho ds \geq 1$ for any locally rectifiable curve $\gamma \in \Gamma$. The extremal length $\lambda(\Gamma)$ of Γ is defined in terms of modulus as follows:

$$\lambda(\Gamma) = (\mathrm{mod}(\Gamma))^{\frac{1}{1-n}}.$$

Note that in the plane $\mathbb{R}^2$, extremal length is just the reciprocal of modulus. In the plane, the concept of extremal length (or modulus) was introduced by Ahlfors and Beurling in