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Analytic continuation and high energy estimates for the resolvent of the Laplacian on forms on asymptotically hyperbolic spaces

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A B S T R A C T

We prove the analytic continuation of the resolvent of the Laplacian on asymptotically hyperbolic spaces on differential forms, including high energy estimates in strips. This is achieved by placing the spectral family of the Laplacian within the framework developed, and applied to scalar problems, by the author recently, roughly by extending the problem across the boundary of the compactification of the asymptotically hyperbolic space in a suitable manner. The main novelty is that the non-scalar nature of the operator is dealt with by relating it to a problem on an asymptotically Minkowski space to motivate the choice of the extension across the conformal boundary.

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1. Introduction

Suppose that (X, g) is an n-dimensional asymptotically hyperbolic space with an even metric in the sense of Guillarmou [12]. That is, g is Riemannian on X, X has a compactification $\overline{X}$ with boundary defining function x, and there is a neighborhood $U = [0, \epsilon)_x \times \partial X$ of ∂X on which g is of the warped product form $\frac{dx^2+h}{x^2}$, with $h = h(x, .)$ a smooth family of symmetric 2-cotensors on ∂X whose Taylor series at $x = 0$ is even, and $h(0, .)$ is positive definite. We refer to [12] for a more geometric version, and to Graham and Lee [11, Section 5] for how to put an arbitrary asymptotically hyperbolic metric, i.e. one for which $x^2 g$ is Riemannian on $\overline{X}$ and $|dx|_{x^2 g} = 1$ at $x = 0$, into a warped product form. We write $\overline{X}_{\text{even}}$ for $\overline{X}$ equipped with the even smooth structure, i.e. using coordinate charts $[0, \epsilon^2)_\mu \times O$, O a coordinate chart in ∂X, in the product decomposition above, where $\mu = x^2$. (So a $\mathcal{C}^\infty$ function on $\overline{X}$ is in $\mathcal{C}^\infty(\overline{X}_{\text{even}})$ if and only if its Taylor series has only even terms at $x = 0$.)

Let Δ_k denote the Laplacian on k-forms on the complete Riemannian manifold (X, g). Thus, Δ_k with domain $\mathcal{C}_c^\infty(X; \Lambda^k X)$ is essentially self-adjoint, and is indeed non-negative, so in particular $(\Delta_k - \lambda)^{-1}$ exists for $\lambda \in \mathbb{C} \setminus [0, \infty)$. We show that

Theorem 1.1. *The operators*

$$\delta d(\Delta_k - \sigma^2 - (n - 2k - 1)^2/4)^{-1},\ d\delta(\Delta_k - \sigma^2 - (n - 2k + 1)^2/4)^{-1}$$

have a meromorphic continuation from $\operatorname{Im}\sigma \gg 1$ *to* $\mathbb{C}$ *with finite rank poles and with non-trapping, resp. mildly trapping, high energy estimates in strips* $|\operatorname{Im}\sigma| < C$ *if* g *is a non-trapping, resp. mildly trapping, metric.*

Here recall that g non-trapping means that all geodesics approach ∂X as the time parameter goes to $\pm\infty$, while mildly trapping, defined in [21, Section 2], is an analytic assumption on a model problem near the trapping (roughly polynomial bounds for the model resolvent) and the nearby bicharacteristic flow; we recall this briefly at the end of Section 4. Non-trapping high-energy estimates mean that for all $C_0 > 0$ and s with $s + 3/2 > C_0$ there is $C > 0$ and $R > 0$ such that

$$\begin{aligned}
&\|\delta d(\Delta_k - \sigma^2 - (n - 2k - 1)^2/4)^{-1}\|_{\mathcal{L}(\mathcal{Y}^{s+1}_{\delta d}, \mathcal{X}^s_{\delta d})} \leq C|\sigma|,\\
&\|d\delta(\Delta_k - \sigma^2 - (n - 2k + 1)^2/4)^{-1}\|_{\mathcal{L}(\mathcal{Y}^{s+1}_{d\delta}, \mathcal{X}^s_{d\delta})} \leq C|\sigma|, \qquad (1.1)\\
&|\operatorname{Im}\sigma| < C_0,\ |\operatorname{Re}\sigma| > R,
\end{aligned}$$

where the norms are on suitable (high-energy) Sobolev spaces, namely

$$\begin{aligned}
\mathcal{X}^s_{\delta d} &= x^{-\imath\sigma + (n-2k-1)/2} H^s_{|\sigma|^{-1}}(\overline{X}_{\text{even}}; \Lambda^k \overline{X}_{\text{even}}),\\
\mathcal{Y}^{s+1}_{\delta d} &= x^{-\imath\sigma + (n-2k-1)/2+2} H^{s+1}_{|\sigma|^{-1}}(\overline{X}_{\text{even}}; \Lambda^k \overline{X}_{\text{even}}),
\end{aligned}$$