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# A duality map for quantum cluster varieties from surfaces

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We define a canonical map from a certain space of laminations on a punctured surface into the quantized algebra of functions on a cluster variety. We show that this map satisfies a number of special properties conjectured by Fock and Goncharov. Our construction is based on the "quantum trace" map introduced by Bonahon and Wong.


**Contents**



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## 1. Introduction

### *1.1. Fock and Goncharov's duality map*

In their seminal paper [4], Fock and Goncharov associated, to a punctured surface $S$, a pair of moduli spaces denoted $\mathcal{A}_{SL_2,S}$ and $\mathcal{X}_{PGL_2,S}$. These spaces are closely related to moduli spaces of $SL_2$- and $PGL_2$-local systems, respectively. Fock and Goncharov showed that certain ideas from Teichmüller theory can be understood in terms of these objects [4].

One of the main results of [4] (see also [5]) was the existence of a duality between the spaces $\mathcal{A}_{SL_2,S}$ and $\mathcal{X}_{PGL_2,S}$. More precisely, Fock and Goncharov defined a space $\mathcal{A}_{SL_2,S}(\mathbb{Z}^t)$ which is a tropicalization of $\mathcal{A}_{SL_2,S}$, and they constructed a canonical map

$$\mathbb{I} : \mathcal{A}_{SL_2,S}(\mathbb{Z}^t) \to \mathbb{Q}(\mathcal{X}_{PGL_2,S})$$

from this space into the algebra of rational functions on $\mathcal{X}_{PGL_2,S}$. The spaces $\mathcal{A}_{SL_2,S}(\mathbb{Z}^t)$ and $\mathcal{X}_{PGL_2,S}$ admit natural coordinates $\{a_i\}_{i\in I}$ and $\{X_i\}_{i\in I}$, parametrized by the set $I$ of edges of an ideal triangulation of $S$. If we number these edges so that $I = \{1, \dots, n\}$, then we have the following result.

**Theorem 1.1** *([4], Theorem 12.2). The canonical functions $\mathbb{I}(\ell)$ defined by the above construction satisfy the following properties:*

1. *For any choice of ideal triangulation, $\mathbb{I}(\ell)$ is a Laurent polynomial in the coordinates $X_i$ with highest term $X_1^{a_1} \dots X_n^{a_n}$ where $a_i$ is the coordinate of $\ell$ associated to the edge $i$.*
2. *The coefficients of the Laurent polynomial $\mathbb{I}(\ell)$ are positive integers.*
3. *For any $\ell$, $\ell' \in \mathcal{A}_{SL_2,S}(\mathbb{Z}^t)$, we have*

$$\mathbb{I}(\ell)\mathbb{I}(\ell') = \sum_{\ell''\in\mathcal{A}_{SL_2,S}(\mathbb{Z}^t)} c(\ell,\ell';\ell'')\mathbb{I}(\ell'')$$

*where $c(\ell, \ell'; \ell'')$ are nonnegative integers and only finitely many terms are nonzero.*