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Skew-signings of positive weighted digraphs

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Abstract. An *arc-weighted digraph* is a pair (D, ω) where D is a digraph and ω is an *arc-weight function* that assigns to each arc uv of D a nonzero real number $\omega(uv)$. Given an arc-weighted digraph (D, ω) with vertices $v_1, \dots, v_n$, the *weighted adjacency matrix* of (D, ω) is defined as the $n \times n$ matrix $A(D, \omega) = [a_{ij}]$ where $a_{ij} = \omega(v_i v_j)$ if $v_i v_j$ is an arc of D, and 0 otherwise. Let (D, ω) be a positive arc-weighted digraph and assume that D is loopless and symmetric. A *skew-signing* of (D, ω) is an arc-weight function ω' such that $\omega'(uv) = \pm\omega(uv)$ and $\omega'(uv)\omega'(vu) < 0$ for every arc uv of D. In this paper, we give necessary and sufficient conditions under which the characteristic polynomial of $A(D, \omega')$ is the same for all skew-signings ω' of (D, ω). Our main theorem generalizes a result of Cavers et al. (2012) about skew-adjacency matrices of graphs.

Keywords: Arc-weighted digraphs; Skew-signing of a digraph; Weighted adjacency matrix

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1. INTRODUCTION

A *directed graph* or, more simply, a *digraph* D is a pair $D = (V, E)$ where V is a set of *vertices* and E is a set of ordered pairs of vertices called *arcs*. For $u, v \in V$, the arc $a = (u, v)$

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of D is denoted by uv. An arc of the form uu is called a *loop* of D. A *loopless digraph* is one containing no loops. A *symmetric* digraph is a digraph such that if uv is an arc then vu is also an arc. Given a symmetric digraph $D = (V, E)$ and a subdigraph $H = (W, F)$ of D, we denote by H^* the subdigraph of D whose vertex set is W and arc set is $\{vu : uv \in F\}$.

Let G be a simple undirected and finite graph. An *orientation* of G is an assignment of a direction to each edge of G so that we obtain a directed graph $\overrightarrow{G}$. Let $\overrightarrow{G}$ be an orientation of G. With respect to a labeling $v_1, \dots, v_n$ of the vertices of G, the *skew-adjacency* matrix of $\overrightarrow{G}$ is the $n \times n$ real skew-symmetric matrix $S(\overrightarrow{G}) = [s_{ij}]$, where $s_{ij} = 1$ and $s_{ji} = -1$ if $v_i v_j$ is an arc of $\overrightarrow{G}$, otherwise $s_{ij} = s_{ji} = 0$. The *skew-characteristic polynomial* of $\overrightarrow{G}$ is defined as the characteristic polynomial of $S(\overrightarrow{G})$. This definition is correct because skew-adjacency matrices of $\overrightarrow{G}$ with respect to different labelings are permutationally similar and so have the same characteristic polynomial.

There are several recent works about skew-characteristic polynomials of oriented graphs, one can see for example [1,3–6,10]. An open problem is to find the number of possible orientations with distinct skew-characteristic polynomials of a given graph G. In particular it is of interest to know whether all orientations of a graph G can have the same skew-characteristic polynomial. The following theorem, obtained by Cavers et al. [3] gives an answer to this question.

Theorem 1.1. *The orientations of a graph G all have the same skew-characteristic polynomial if and only if G has no cycles of even length.*

A similar result to this theorem was obtained by Liu and Zhang [7]. They proved that all orientations of a graph G have the same permanental polynomial if and only if G has no cycles of even length.

In this work, we will extend Theorem 1.1 to positive weighted loopless and symmetric digraphs (which we abbreviate to *pwls-digraphs*). *An arc-weighted digraph* or more simply a *weighted digraph* is a pair (D, ω) where D is a digraph and ω is a *arc-weight function* that assigns to each arc uv of D a nonzero real number $\omega(uv)$, called *the weight* of the arc uv. Let (D, ω) be a weighted digraph with vertices $v_1, \dots, v_n$. The weighted adjacency matrix of (D, ω) is defined as the $n \times n$ matrix $A(D, \omega) = [a_{ij}]$ where $a_{ij} = \omega(v_i v_j)$, if $v_i v_j$ is an arc of D and 0 otherwise. Let (D, ω) be a pwls-digraph. A skew-signing of (D, ω) is an arc-weight function ω' such that $\omega'(uv) = \pm\omega'(uv)$ and $\omega'(uv)\omega'(vu) < 0$ for every arc uv of D.

Our aim is to characterize the pwls-digraphs (D, ω) for which the characteristic polynomial of $A(D, \omega')$ is the same for all skew-signings ω' of (D, ω). This characterization involves directed cycles in D. Recall that a *directed cycle* of length $t > 0$ is a digraph with vertex set $\{v_1, v_2, \dots, v_t\}$ and arcs $v_1 v_2, \dots, v_{t-1} v_t, v_t v_1$. Throughout this paper, we use the term "cycle" to refer to a "directed cycle" in a digraph. A cycle of length $t = 2$ is called a *digon*. A cycle is *odd* (resp. *even*) if its length is odd (resp. even). Our main result is the following theorem.

Theorem 1.2. *Let (D, ω) be a pwls-digraph. Then the following statements are equivalent:*

(i) *The characteristic polynomial of (D, ω') is the same for all skew-signings ω' of (D, ω).*
(ii) *D has no even cycles of length more than 2 and $A(D, \omega) = \Delta^{-1} S \Delta$ where S is a nonnegative symmetric matrix with zero diagonal and Δ is a diagonal matrix with positive diagonal entries.*