ARAB JOURNAL OF MATHEMATICAL SCIENCES

Arab J Math Sci ▮ (▮) (▮▮▮▮) ▮▮▮–▮▮▮

Best proximity pair and fixed point results for noncyclic mappings in modular spaces

KARIM CHAIRA, SAMIH LAZAIZ*

Laboratory of Algebra Analysis and Applications, Department of Mathematics and Computer Sciences, Faculty of Sciences Ben M'Sik, Hassan II University of Casablanca, BP 7955, Sidi Othman, Casablanca, Morocco

Received 17 July 2017; revised 28 January 2018; accepted 8 February 2018
Available online xxxx

Abstract. In this paper, we formulate best proximity pair theorems for noncyclic relatively ρ-nonexpansive mappings in modular spaces in the setting of proximal ρ-admissible sets. As a companion result, we establish a best proximity pair theorem for pointwise noncyclic contractions in modular spaces. To that end, we provide some examples throughout the paper to illustrate the validity of the obtained results.

Keywords: Best proximity pair; Modular spaces; Relatively ρ-nonexpansive mappings; ρ-admissible sets; ρ-normal structure

Mathematics Subject Classification: 47H09; 41A65

1. INTRODUCTION

Let X be an arbitrary vector space.

1. A function $\rho : X \to [0, \infty]$ is called a modular on X if for arbitrary $x, y \in X$,
   (a) $\rho(x) = 0$ if and only if $x = 0$,
   (b) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,
   (c) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.

* Corresponding author.
E-mail addresses: chaira_karim@yahoo.fr (K. Chaira), samih.lazaiz@gmail.com (S. Lazaiz).
Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

https://doi.org/10.1016/j.ajmsc.2018.02.002

If (c) is replaced by (c)': $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$, we say ρ is convex modular.

2. A modular ρ defines a corresponding modular space, i.e. the vector space X_ρ given by

$$X_\rho = \{x \in X \ : \ \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

X_ρ is a linear subspace of X.

The relevance of a best proximity pair, in a couple of non-empty, disjoint subsets A and B of a modular space, is to act as a substitute in the absence of a fixed point. It is also used to provide optimal solutions to the problem of best approximation between two sets.

Eldred, Kirk and Veeramani [7] established the existence of a best proximity pair for noncyclic relatively nonexpansive mappings by using a geometric notion of proximal normal structure in the setting of Banach spaces. The work of the afore-mentioned authors generalizes the notion of normal structure introduced by Milman and Brodskii [6]. Recently, Sankar and Veeramani established the existence and uniqueness of a best proximity pair for noncyclic contraction maps as stated in [18]. Similar results in [1] were discussed by Taghafi and Shahzad who proved the existence of a best proximity pair for a cyclic contraction map in a reflexive Banach space. For other related results, we refer the reader to [1–5,9,10,21,22].

In this paper, we generalize the notion of proximal ρ-normal structure for a ρ-admissible pair (A, B) in modular spaces. We also show that if A and B are proximal ρ-admissible sets, and if the pair (A, B) has proximal ρ-normal structure, then every noncyclic relatively ρ-nonexpansive map has a best proximity pair. As a companion result, we show the existence and uniqueness of a best proximity pair theorem for pointwise noncyclic contractions in the setting of modular spaces.

2. Preliminaries

To describe our results, we need to review some basic definitions and notions related to modular spaces, such as those formulated by Musielak and Orlicz [20]. For further details, we refer the reader to [12,14,16,19]

Definition 1. Let X_ρ be a modular space.

1. We say that (x_n) is ρ-convergent to x and write $x_n \to x\ (\rho)$ if and only if $\rho\,(x_n - x) \to 0$.
2. A sequence (x_n), where $x_n \in X_\rho$, is called ρ-Cauchy if $\rho\,(x_n - x_m) \to 0$ as $n, m \to \infty$.
3. We say that X_ρ is ρ-complete if and only if any ρ-Cauchy sequence in X_ρ is ρ-convergent.
4. A set $C \subset X_\rho$ is called ρ-closed if for any sequence (x_n) of C, the convergence $x_n \to x(\rho)$ implies that x belongs to C.
5. A set $C \subset X_\rho$ is called ρ-sequentially-compact if for any sequence (x_n) of C, there exists a convergent subsequence $\left(x_{n_k}\right)_k$ of (x_n) such that $x_{n_k} \to x(\rho)$ in C.
6. A set $C \subset X_\rho$ is called ρ-bounded if $\sup\{\rho\,(x - y) : x, y \in C\} < \infty$.
7. We will say that ρ satisfies the Fatou property if

$$\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$$

whenever $x_n \to x\ (\rho)$.

Please cite this article in press as: K. Chaira, S. Lazaiz, Best proximity pair and fixed point results for noncyclic mappings in modular spaces, Arab Journal of Mathematical Sciences (2018), https://doi.org/10.1016/j.ajmsc.2018.02.002.