ARAB JOURNAL OF
MATHEMATICAL SCIENCES

Arab J Math Sci ■ (■) (■■■■) ■■■–■■■

Some results and examples of the biharmonic maps with potential

ABDELKADER ZAGANE[a], SEDDIK OUAKKAS[b,*]

[a] Department of mathematics - University of Mascara, Algeria
[b] Laboratory of Geometry, Analysis, Control and Applications, University of Saida, Algeria

Received 24 December 2017; revised 21 March 2018; accepted 4 April 2018
Available online xxxx

Abstract. In this paper, we will study the class of biharmonic maps with potential, in the particular case represented by conformal maps between equidimensional manifolds. Some examples are constructed in particular cases (Euclidean space and sphere).

Keywords: Harmonic map with potential; Biharmonic map with potential; Conformal maps

Mathematics Subject Classification: 31B30; 53C25; 58E20; 58E30

1. INTRODUCTION

The notion of harmonic maps with potential was first suggested by A. Ratto and A. Fardoun (see [5] and [9]). Let (M^m, g) and (N^n, h) be Riemannian manifolds, H a smooth function on N, and let $\phi : M \to N$ be a smooth map. We consider the following energy functional

$$E_H(\phi) = \int_K (e(\phi) - H(\phi))\, dv_g \tag{1}$$

for any compact subset $K \subset M$. The Euler–Lagrange equation of $E_H(\phi)$ is

$$\tau_H(\phi) = \tau(\phi) + \left(grad^N H\right) \circ \phi = 0, \tag{2}$$

* Corresponding author.
E-mail addresses: a.zaagane@yahoo.com (A. Zagane), seddik.ouakkas@univ-saida.dz (S. Ouakkas).
Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

https://doi.org/10.1016/j.ajmsc.2018.04.001

where $\tau(\phi) = Tr_g \nabla d\phi$ is the tension field of ϕ. The smooth solutions of (2) will be called harmonic maps with potential H. One can refer to [3], [5] and [9] for background on harmonic maps with potential. In [4], the authors calculate the second variation for harmonic maps with potential and they introduce the notion of biharmonic maps with potential. In this paper, we will recalculate the second variation of the H-energy and the first variation of the H-bi-energy (Theorems 1 and 2). As the second result we give the relation between $\tau_{2,H}(\phi)$ and $\tau_2(\phi)$ (Theorem 3) where we study the case of the identity map (Corollary 2 and Theorem 2) and we construct some examples of biharmonic with a potential. Finally, we study the particular of the conformal maps between equidimensional manifolds (Theorems 5 and 6).

2. The second variation of the H-energy functional

Let $\phi : (M^m, g) \to (N^n, h)$ be a harmonic map with potential H between Riemannian manifolds. By a two parameter variation we mean a smooth map $\Phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \longrightarrow N$ defined by $\Phi(x, t, s) = \phi_{t,s}(x)$, such that $\phi_{0,0} = \phi$. Its variation vector fields are the vector fields v, w along ϕ defined by

$$v = \left.\frac{\partial \phi_{t,s}}{\partial t}\right|_{t=s=0}$$

and

$$w = \left.\frac{\partial \phi_{t,s}}{\partial s}\right|_{t=s=0}.$$

Now suppose that M is compact and let ∇^{ϕ} denote the pull-back connection on $\phi^{-1}TN$. By the Leibniz rule,

$$\left(\nabla^{\phi}\right)^2_{X,Y} v = \nabla^{\phi}_X \nabla^{\phi}_Y v - \nabla^{\phi}_{\nabla^M_X Y} v$$

for any $X, Y \in \Gamma(TM)$ and $v \in \Gamma\left(\phi^{-1}TN\right)$. On taking the trace we obtain

$$Tr_g \left(\nabla^{\phi}\right)^2 v = \nabla^{\phi}_{e_i} \nabla^{\phi}_{e_i} v - \nabla^{\phi}_{\nabla^M_{e_i} e_i} v,$$

where $\{e_i\}_{1 \le i \le m}$ is an orthonormal frame on M and where we sum over repeated indices. Under the notation above we have the following :

Theorem 1 (*The Second Variation Formula*)**.** *Let $\phi : (M^n, g) \to (N^n, h)$ be a harmonic map with potential H and suppose that M is compact, we have*

$$\frac{\partial^2}{\partial t \partial s} E_H\left(\phi_{t,s}\right)\Big|_{t=s=0} = \int_M h\left(J^{\phi}_H(v), w\right) dv_g, \tag{3}$$

where $J^{\phi}_H(v) \in \Gamma\left(\phi^{-1}TN\right)$ is given by

$$J^{\phi}_H(v) = -Tr_g \left(\nabla^{\phi}\right)^2 v - Tr_g R^N(v, d\phi) d\phi - \left(\nabla^N_v grad^N H\right) \circ \phi. \tag{4}$$

Proof of Theorem 1. Let $\Phi : M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \longrightarrow N$, $(x, t, s) \longmapsto \Phi(x, t, s) = \phi_{t,s}(x)$ be a smooth variation of ϕ with variation vector fields v and w. Let ∇^{Φ} denote the pull-back connection on $\Phi^{-1}TN$, a vector bundle over $M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$. Note that

$$\left[\frac{\partial}{\partial t}, X\right] = \left[\frac{\partial}{\partial s}, X\right] = \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = 0$$

Please cite this article in press as: A. Zagane, S. Ouakkas, Some results and examples of the biharmonic maps with potential, Arab Journal of Mathematical Sciences (2018), https://doi.org/10.1016/j.ajmsc.2018.04.001.