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On the structure of conservation laws of (3+1)-dimensional wave equation

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Abstract. In this paper, a (3+1)-dimensional wave equation is studied from the point of view of Lie's theory in partial differential equations including conservation laws. The symmetry operators are determined to find the reduced form of the considered equation. The non-local conservation theorems and multipliers approach are performed on the (3+1)-dimensional wave equation. We obtain conservation laws by using five methods, such as direct method, Noether's method, extended Noether's method, Ibragimov's method; and finally we can derive infinitely many conservation laws from a known conservation law viewed as the last method. We also derive some exact solutions using some conservation laws Anco and Bluman (2002).

Keywords: Wave equation; Conservation laws; Lie symmetry; Direct method; Noether's theorem; Boyer's formulation

Mathematics Subject Classification: 76M60; 70H33; 35J05

1. Introduction

Many partial differential equations (PDEs) of physical importance have not a general theory for finding solutions. While there is no existing general theory for solving such equations the methods of point transformations are a powerful tool. One of the most useful

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point transformations are those which form a continuous group. Lie classical symmetries admitted by a system of PDEs are useful for finding invariant solutions [3,4]. The classical symmetry method for differential equations (DEs) is based on Lie group symmetries. Lie symmetries provide a powerful and systematic tool for analysis of PDEs. For instance, the method of reduction of variables via Lie point symmetries is an extremely useful technique for simplifying or solving PDE's.

The symmetry group of a system of DEs transforms solutions of the system to other solutions of the system. Another important class of symmetries that generalize contact transformations is the class of higher-order symmetries.

A PDEs system can be considered not only as itself but together with its prolongations to all orders. The transformations that preserve the contact system in the infinite jet space and leave the infinite prolongation of the system invariant are called higher-order symmetries; their calculation and possible outcomes upon the wave equation are realized in section two.

A conservation law of a given DEs system is a divergence expression that vanishes on all solutions of the DEs system. In the study of systems of DEs, the concept of a conservation law plays significant roles, not only in obtaining in-depth understanding of physical properties of various systems, but also in the construction of their exact solutions. They describe physical conserved quantities such as mass, energy, momentum and angular momentum, as well as charge and other constants of motion. They are important for investigating integrability and linearization mappings and for establishing existence and uniqueness of solutions. They are also used in the analysis of stability and global behavior of solutions [5,15,8,14]. This work is organized as follows: in Section 3, the conservation laws associated to the wave equation are computed via direct method and other methods. Section 4 is devoted to the construction of exact solutions by utilizing known conservation laws. Finally the conclusions are presented in Section 5.

The general wave equation is an important second-order nonlinear PDE

$$u_{tt} - (f(u)u_x)_x - (g(u)u_y)_y - (h(u)u_z)_z = 0, \tag{1}$$

for the description of waves as they occur in physics such as sound waves, light waves and water waves. It arises in fields like acoustics, electromagnetics, and fluid dynamics. In this paper we study Eq. (1) for $f(u) = g(u) = h(u) = 1$ which takes the form

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0. \tag{2}$$

Then the wide range of solution with Lie symmetry method and conservation are given.

2. Lie symmetries of the equation

It is known that to find exact solutions of PDEs is always one of the central themes in mathematics and physics. Some of the most important methods for finding exact solutions of PDEs are the Lie symmetry analysis [17,19,18,6,12,11].

First of all, let us consider a one-parameter Lie group of infinitesimal transformation:

$$\begin{aligned}
t &= t + \xi_1(t, x, y, z, u) + O(\varepsilon^2),\\
x &= x + \xi_2(t, x, y, z, u) + O(\varepsilon^2),\\
y &= y + \xi_3(t, x, y, z, u) + O(\varepsilon^2),\\
z &= z + \xi_4(t, x, y, z, u) + O(\varepsilon^2),\\
u &= u + \phi(t, x, y, z, u) + O(\varepsilon^2)
\end{aligned} \tag{3}$$