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On some higher order boundary value problems at resonance with integral boundary conditions

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Abstract. This paper investigates the existence of solutions for higher-order multipoint boundary value problems at resonance. We obtain existence results by using coincidence degree arguments.

Keywords: Multipoint boundary value problems; Resonance; Integral boundary conditions; Coincidence degree; Higher-order

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1. INTRODUCTION

In this article, we consider the following higher-order boundary value problems:

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \ldots, x^{(n-1)}(t)\right) + e(t) \tag{1.1}$$

$$x^{(n-1)}(0) = \alpha x(\xi), \quad x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(1) = \int_0^1 x(s)dg(s) \tag{1.2}$$

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where $\alpha \geq 0, 0 < \xi < 1$, $f : [0, 1] \times \Re^n \to \Re$ is a continuous function, $e : [0, 1] \to \Re$ is a function in $L^1[0, 1]$ and $g : [0, 1] \to [0, \infty)$ is a nondecreasing function with $g(0) = 0$ and $g(1) = 1$. The integral in (1.2) is a Riemann–Stieltjes integral.

Multipoint boundary value problems of ordinary differential equations arise in many areas of Physics, Engineering and Applied Mathematics. In particular, integral boundary conditions are encountered in various applications such as population dynamics, blood flow models and cellular systems. In recent years, higher-order boundary value problems have appeared in many papers, for example, see [1–7] and the references therein. To the best of our knowledge, the corresponding problem for higher-order ordinary differential equations with integral boundary conditions at resonance has received little attention.

The boundary value problem (1.1)–(1.2) is called a problem at resonance if $Lx = x^{(n)}(t) = 0$ has non-trivial solutions under boundary condition (1.2), that is, when $dimKerL \geq 1$. When $KerL = 0$, the differential operator L is invertible. In this case, the problem is at non-resonance. The remainder of this paper is organized as follows. In Section 2 we provide some results and lemmas which are important in stating and proving the main existence theorems. In Section 3, the statement and proof of the main existence results are provided.

2. Preliminaries

In this section we present some preliminaries that will be used in the subsequent sections. Let X and Z be real Banach spaces and let $L : domL \subset X \to Z$ be a linear Fredholm operator of index zero. Let $P : X \to X$ and $Q : Z \to Z$ be continuous projections such that $ImP = KerL$, $KerQ = ImL$ and $X = KerL \oplus KerP$, $Z = ImL \oplus ImQ$. It follows that $L|_{domL \cap KerP} : domL \cap KerP \to ImL$ is invertible. We denote this inverse by K_p.

If Ω is an open bounded subset of X such that $domL \cap \Omega \neq \emptyset$, then the map $N : X \to Z$ is called $L -$ compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact, with P and Q as above.

In what follows we shall use the classical spaces $C^{n-1}[0, 1]$, $L^1[0, 1]$. For $x \in C^{n-1}[0, 1]$, we use the norm $\|x\|_\infty = \max_{t \in [0,1]} |x(t)|$ and $\|x\| = max\{\|x\|_\infty, \|x'\|_\infty, \ldots, \|x^{n-1}\|_\infty\}$. We denote the norm in $L^1[0, 1]$ by $\| \cdot \|_1$. We use the Sobolev space $W^{n,1}(0, 1)$ defined by $W^{n,1}(0, 1) = \{x : [0, 1] \to \Re | x, x', \ldots, x^{(n-1)}$ are absolutely continuous on [0,1] with $x^{(n)} \in L^1[0, 1]\}$. Let $X = C^{n-1}[0, 1]$, $Z = L^1[0, 1]$. L is the linear operator from $domL \subset X \to Z$ with $domL = \{x \in W^{n,1}(0, 1) : x^{(n-1)}(0) = \alpha x(\xi), x'(0) = x''(0) = \cdots = x^{n-2}(0) = 0, x(1) = \int_0^1 x(s)dg(s)\}$. We define $L : domL \subset X \to Z$ by $Lx = x^{(n)}(t)$ and $N : X \to Z$ by $Nx = f\left(t, x(t), x'(t), \ldots, x^{n-1}(t)\right) + e(t)$. Then, the boundary value problem (1.1)–(1.2) becomes

$$Lx = Nx$$

We shall discuss existence results for (1.1)–(1.2) in the following cases.
Case 1: $\alpha = \frac{(n-1)!}{\xi^{n-1}}$, $\int_0^1 s^{n-1}dg(s) = 1$, $\int_0^1 s^n dg(s) \neq 1$, $g(1) = 1$, $g(0) = 0$
Case 2: $\alpha = 0$, $\int_0^1 s^n dg(s) \neq 1$, $g(1) = 1$, $g(0) = 0$

3. Existence results

We shall use the following fixed point theorem of Mawhin [8] to obtain our existence results.