Bull. Sci. math. 146 (2018) 1–32

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

Twisted argyle quivers and Higgs bundles

Steven Rayan *, Evan Sundbo

Department of Mathematics & Statistics, McLean Hall, University of Saskatchewan, Saskatoon, SK, S7N 5E6, Canada

ARTICLE INFO

Article history:
Received 12 March 2018
Available online 21 March 2018

MSC:
14D20
14H60
14L24
16G20

Keywords:
Quiver variety
Argyle quiver
Higgs bundle
Moduli space
Stability

ABSTRACT

Ordinarily, quiver varieties are constructed as moduli spaces of quiver representations in the category of vector spaces. It is also natural to consider quiver representations in a richer category, namely that of vector bundles on some complex variety equipped with a fixed sheaf that twists the morphisms. Representations of A-type quivers in this twisted category — known in the literature as “holomorphic chains” — have practical use in questions concerning the topology of the moduli space of Higgs bundles. In that problem, the variety is a Riemann surface of genus at least 2, and the twist is its canonical line bundle. We extend the treatment of twisted A-type quiver representations to any genus using the Hitchin stability condition induced by Higgs bundles and computing their deformation theory. We then focus in particular on so-called “argyle quivers”, where the rank labelling alternates between 1 and integers $r_i \geq 1$. We give explicit geometric identifications of moduli spaces of twisted representations of argyle quivers on $\mathbb{P}^1$ using invariant theory for a non-reductive action via Euclidean reduction on polynomials. This leads to a stratification of the moduli space by change of bundle type, which we identify with “collision manifolds” of invariant zeroes of polynomials. We also relate the present work to Bradlow–Daskalopoulos stability and Thaddeus’ pullback maps to stable tuples. We apply our results to computing $\mathbb{Q}$-Betti numbers of low-rank twisted Higgs bundle moduli spaces on $\mathbb{P}^1$, where the Higgs fields take values in an arbitrary ample

* Corresponding author.
E-mail addresses: rayan@math.usask.ca (S. Rayan), ejs383@mail.usask.ca (E. Sundbo).

https://doi.org/10.1016/j.bulsci.2018.03.003

line bundle. Our results agree with conjectural Poincaré series arising from the ADHM recursion formula.

1. Introduction

By now, quiver varieties are household names in mathematics, particularly when it comes to representation theory and geometry. Most often they are formed by labelling the nodes of a directed graph with nonnegative integers and then considering linear representations up to isomorphism. Here, we are referring to representations in the category of vector spaces: to each node, we assign a vector space (over $\mathbb{C}$ or some other algebraically-closed field) of the prescribed dimension, and a corresponding linear map to each arrow. Another common construction of quiver varieties is the Nakajima construction [27,28]. This also uses the category of vector spaces, but each arrow in the initial quiver is "doubled" by adding an arrow with the opposite orientation. These are interpreted as cotangent directions to the space of representations of the original quiver. The quiver variety is then constructed as a hyperkähler quotient [23], and thus provides an important class of examples of Calabi–Yau manifolds. Broadly speaking, quiver varieties have applications to toric geometry, vertex algebras, noncommutative geometry, integrable systems, and gauge theories.

Quiver varieties can also be constructed in other categories. The natural generalization of the category of vector spaces is the category $\mathrm{Bun}(X)$ of vector bundles on a fixed complex variety X. Here, we label each node with two integers, r_i and d_i, where $r_i \geq 0$. The r_i numbers specify the ranks of the corresponding vector bundles while the d_i's fix their respective degrees. If one node is represented by U_i and another is represented by U_j and there is an arrow between them, then we assign to that arrow a vector bundle morphism in $\mathrm{Hom}(U_i, U_j)$. One of the main applications of working in this setting is to the study of Higgs bundles. A *Higgs bundle* is a vector bundle $E \to X$ together with a regular $\mathcal{O}_X$-linear map $\Phi : E \to E \otimes \omega_x$ satisfying the integrability condition $\Phi \wedge \Phi = 0$, where ω_X is the canonical line bundle of X [21,39]. If we consider a "twisted" category $\mathrm{Bun}(X, \omega_X)$ where the morphisms are twisted by ω_X, we can pose the following alternative but equivalent definition: A Higgs bundle is a representation of rank r and degree d of a single node, labelled r, d, with a loop.

What makes this useful and not just a "rebranding" is that the moduli space of Higgs bundles on X comes with a natural linear action of $\mathbb{C}^\times$ that rescales Φ. The cohomology of the moduli space localizes to the components of the fixed-point set [21,38,14,20,13], which themselves are positive-dimensional due to the noncompactness of the moduli space. The components are indexed by partitions of r and d into some number of parts that are strung out into an A-type quiver. The fixed points themselves are representations of these quivers in $\mathrm{Bun}(X, \omega_X)$, and so the problem of computing topological invariants