Bull. Sci. math. 144 (2018) 1–38

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

Time fractional linear problems on $L^2(\mathbb{R}^d)$

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A R T I C L E I N F O

Article history:
Received 4 October 2017
Available online 31 January 2018

MSC:
35C15
35E99

Keywords:
Fractional derivatives
Time fractional PDEs
Regularizing effect
Dissipative inequalities
Fractional Schrödinger equation
Fractional transport equation

A B S T R A C T

In this paper, we propose a theory for linear time fractional PDEs on $L^2(\mathbb{R}^d)$, with two time parameters. The order of the time derivatives under consideration is less than 1. We study well-posedness, regularizing effects and dissipative properties. Regarding regularizing effects, we describe quite precisely the equations that have this effect or not. We highlight that, in purely fractional settings, the regularizing effect acts always only up to finite order; unlike to the standard case. Also, we investigate the properties of the three time variables solution operator generated by these PDEs.

1. Introduction

Time fractional differential equations have been the subject of many researches in the recent years, both in terms of applications ([8], [13], [3]) and mathematical studies ([12], [9]).

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https://doi.org/10.1016/j.bulsci.2018.01.002
0007-4497/

For problems of the form

$$\mathbf{D}^{\alpha}_{0,t} = P(D)u, \quad u(0) = 0, \tag{1.1}$$

where $u = u(x,t)$ and $P(D)$ is a differential operator acting on the space variable x, there are well-posedness results coming from the analysis of abstract time fractional equations (see for instance [11], [2], [10]). However, as pointed out in [7], general time fractional problems on $L^2(\mathbb{R}^d)$ are

$$\mathbf{D}^{\alpha}_{\tau,t} u = P(D)u, \quad u(s) = v, \tag{1.2}$$

where $\tau \leq s$. It is shown in [7] that the analysis of (1.2) cannot be reduced to the one of (1.1) when $\tau < s$. So, the aim of this paper is to built a theory for general linear time fractional PDEs on $L^2(\mathbb{R}^d)$ of the form (1.2). We will give quantitive and qualitative results for this problems.

For first order time derivative equations of the form

$$u_t = P(D)u, \tag{1.3}$$

the well posedness is equivalent to the condition

$$\sup_{\xi \in \mathbb{R}^d} \operatorname{Re} P(\xi) < \infty,$$

where P is the symbol of $P(D)$. We refer to subsections 2.1 and 2.3 for notation and definitions.

For time fractional equations of the form (1.1) with $\alpha \in (0,1)$, the above condition is sufficient but no more necessary (see Example 4.3 below).

Roughly speaking, our results state that the critical angle for equation (1.1) is $\frac{\pi}{2}\alpha$ in a similar fashion that $\pi/2$ is the critical angle for (1.3). Indeed, if $|\arg P(\xi)| < \frac{\pi}{2}\alpha$ then the Cauchy problem (1.1) is generally ill-posed. If $|\arg P(\xi)| > \frac{\pi}{2}\alpha$ then the problem is well-posed, has regularizing effects and dissipative properties. If $|\arg P(\xi)| = \frac{\pi}{2}\alpha$ then the problem is well-posed in $L^2(\mathbb{R}^d)$, possesses a kind of asymptotic conservation law (Theorem 4.4) but, has no regularizing and dissipative properties (see Proposition 4.5).

For instance, the *fractional Schrödinger equation*

$$\mathbf{D}^{\alpha}_{0,t} u = -\mathrm{i}\Delta u, \tag{1.4}$$

is hyperbolic for $\alpha = 1$ in the sense that it has a conservation law and no regularizing effect. In this case, $\arg P(\xi) = \pi/2$.

When $\alpha \in (0,1)$, the critical angle is $\frac{\pi}{2}\alpha$. Since $\pi/2 > \frac{\pi}{2}\alpha$, there results that (1.4) has regularizing effect and dissipative properties (see Theorem 6.3). In order to recover a hyperbolic behaviour for $\alpha \in (0,1)$, Equation (1.4) may be modified into