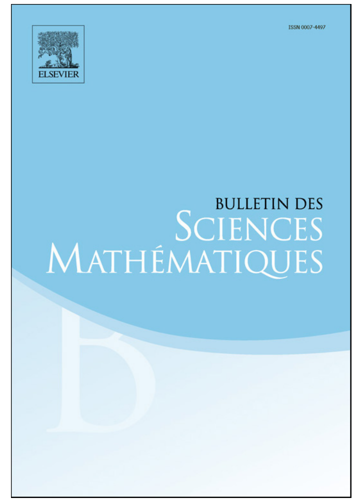

Born–Jordan pseudodifferential operators and the Dirac correspondence: beyond the Groenewold–van Hove Theorem

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BORN–JORDAN PSEUDODIFFERENTIAL OPERATORS AND THE DIRAC CORRESPONDENCE: BEYOND THE GROENEWOLD–VAN HOVE THEOREM

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ABSTRACT. Quantization procedures play an essential role in operator theory, time-frequency analysis and, of course, in quantum mechanics. Roughly speaking the basic idea, due to Dirac, is to associate to any symbol, or observable, $a(x,\xi)$ an operator $\mathrm{Op}(a)$, according to some axioms dictated by physical considerations. This led to the introduction of a variety of quantizations. They all agree when the symbol $a(x,\xi) = f(x)$ depends only on x or $a(x,\xi) = g(\xi)$ depends only on ξ:

$$\mathrm{Op}(f \otimes 1)u = fu, \quad \mathrm{Op}(1 \otimes g)u = \mathcal{F}^{-1}(g\mathcal{F}u)$$

where $\mathcal{F}$ stands for the Fourier transform. Now, Dirac aimed at finding a quantization satisfying, in addition, the key correspondence

$$[\mathrm{Op}(a), \mathrm{Op}(b)] = i\mathrm{Op}(\{a,b\})$$

where $[\,,\,]$ stands for the commutator and $\{\,,\,\}$ for the Poisson brackets, which would represent a tight link between classical and quantum mechanics. Unfortunately, the famous Groenewold–van Hove theorem states that such a quantization does not exist, and indeed most quantization rules satisfy this property only approximately.

In this work we show that the above commutator rule in fact holds for the Born-Jordan quantization, at least for symbols of the type $f(x)+g(\xi)$. Moreover we will prove that, remarkably, this property completely characterizes this quantization rule, making it the quantization which best fits the Dirac dream.

1. Introduction

The theory of pseudodifferential operators has had many avatars since its inception in the mid 1960s; it has developed into a major branch of operator theory since the pioneering work of R. Beals, H. Duistermaat, C. Fefferman, L. Hörmander, J. J. Kohn, R. Melrose, L. Nirenberg, M. A. Shubin, M. E. Taylor, and many others. Its role in non-commutative geometry is also well known [18, 27]. One early precursor, having its origin in quantum mechanics, and which gained its mathematical *lettres de noblesse* only in 1979 following the work of Hörmander [22], is the theory of Weyl operators. It was observed by Stein [31], §75–7.6, that the Weyl pseudodifferential calculus is *uniquely*

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