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Partial differential equations/Game theory

Mean-field games with a major player

Jeux à champ moyen avec agent dominant

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A R T I C L E I N F O

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A B S T R A C T

We introduce and study mathematically a new class of mean-field-game systems of equations. This class of equations allows us to model situations involving one major player (or agent) and a "large" group of "small" players.

R É S U M É

Nous introduisons et étudions mathématiquement une classe nouvelle de jeux à champ moyen. Les systèmes d'équations que nous présentons permettent de modéliser les situations faisant intervenir un joueur dominant et un « grand » groupe de « petits » joueurs.

Version française abrégée

Dans cette note, nous introduisons et étudions une nouvelle classe de modèles mathématiques pour des situations où interviennent un « grand » groupe de « petits » joueurs indistinguables et un joueur dominant. En d'autres termes, nous introduisons et étudions une classe nouvelle de modèles de jeux à champ moyen (MFG en abrégé) avec agent dominant. Nous renvoyons le lecteur à la version anglaise pour une brève introduction aux MFG et pour quelques références de travaux antérieurs.

Les situations que nous considérons ici à savoir les situations avec un agent dominant sont très fréquentes dans les applications en particulier à l'Économie. Ces modèles sont délicats à manipuler et à étudier puisque, entre autres difficultés, tout aléa s'appliquant au joueur dominant induit un aléa commun à tous les joueurs, et la partie purement MFG des systèmes est donc de type « bruit commun ».

Dans les cas où les petits joueurs sont décrits par des variables prenant un nombre fini de valeurs notées $i \in \{1, \dots, k\}$ ($k \geqslant 1$), les systèmes sont de la forme suivante

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$$\begin{cases} \frac{\partial U}{\partial t} + (A(X,y,U,\alpha^*).\nabla_X)U \ +\alpha^*.\nabla_y U - \nu\Delta_y U - \varepsilon\Delta_X U + \\ \qquad +\lambda U = B(X,y,U,\alpha^*) \end{cases} \tag{1}$$

$$\begin{cases} \frac{\partial \varphi}{\partial t} + F(X,y,U,\nabla_y\varphi,\alpha^*) \ +A(X,y,U,\alpha^*)\cdot\nabla_X\varphi + \\ \qquad -\nu\Delta_y\varphi - \varepsilon\Delta_X\varphi + \mu\varphi = 0 \end{cases} \tag{2}$$

$$\alpha^* = \frac{\partial F}{\partial p}(X,y,U,\nabla_y\varphi,\alpha^*) \tag{3}$$

pour $X \in \mathbb{R}^k$, $y \in \mathbb{R}^d$, $t \geqslant 0$. Les notations et les hypothèses sur les coefficients et les données (H, F, A, B) sont détaillées dans la version anglaise.

Le système (1)–(3) est complété par des conditions initiales sur U et φ, à savoir

$$U|_{t=0} = U_0(X,y), \varphi|_{t=0} = \varphi_0(X,y) \ \text{sur} \ \mathbb{R}^k \times \mathbb{R}^d \tag{4}$$

où U_0 et φ_0 sont des fonctions « régulières » données.

Il est à noter que nous avons renversé le sens du temps dans les équations précédentes de façon à simplifier les notations.

Dans la version anglaise, nous donnons quelques exemples de résultats mathématiques concernant la solvabilité (i.e. l'existence et l'unicité de solutions régulières globales ou locales en temps) de tels systèmes, et nous indiquons brièvement l'analogue de ces systèmes dans le cas de variables d'états continues, auquel cas les équations sont alors en dimension infinie.[1]

1. Introduction

In this note, we introduce and study a new class of mathematical models for situations involving a large group of indistinguishable "small" players and one major player. In other words, we introduce and study a new class of mean-field games (MFG in short) models with a major player.

MFG models have been introduced by the authors first in the case without common noise for the players ([6], [7], [8]) and next in the general case with the introduction of the so-called master equation (see [9]), which is in general an infinite-dimensional nonlinear partial differential equation. Let us also recall that the essential structure conditions (monotonicity) and most of the existing mathematical tools have also been introduced by the authors (see [6], [7], [8] and [9]). Let us mention that the particular case of MFG models with no common noise was independently considered in [4] and that some particular cases have been discussed previously in the Economics literature (anonymous games in the discrete time case without common noise, or a heuristic description in a Macroeconomics example in [5]). Finally, since their introduction, there is now a huge scientific literature on MFG concerning the mathematical theory and also many applications to Economics, Finance, Social Sciences, Communication Networks, Engineering Sciences, Computer Sciences, etc. We refer the reader to the online courses [9], to the book [2] (and the references therein) for further information on the mathematical theory.

The issue studied and solved here is of paramount importance in many applicative contexts such as economical ones (for instance). Indeed, MFG models allow one to describe the average (in mean-field sense like in Statistical Mechanics and Physics) behavior of a large group of interacting indistinguishable players. However, in many realistic situations, there is also at least one major or dominant player, and this is precisely this type of situation we address here. Let us mention that we could consider as well a "large group of players" dominating another "large group of players", but we shall restrict ourselves to only one major player for the sake of simplicity. There have been a few attempts in this direction (see [1], [3]). However, we believe that the models and the theory we present is new, in particular because the "MFG part" of the models automatically falls in the delicate category of models with common noise as soon as the dynamics of the major player involve some "noise" (or random terms). Indeed, this noise automatically impacts all the other (small) players and becomes a "common noise" for them. As a consequence, it is not possible, in general, to write meaningful "forward–backward" systems of equations, as is the case for MFG without common noise.

At this point, we also wish to emphasize that our theory is obviously reminiscent of Stackelberg equilibria in Game Theory, an important concept for static or iterated games, which however is delicate to extend to continuous time situations such as the ones we consider. Our models rely on two simplifications, which seem to be well accepted in Economics Theory by now, namely: (i) we only consider feedbacks and (ii) the major player does not anticipate in the way the strategy of the other players is taken into account in its dynamic optimization. These simplifications imply some recursivity features, which, in turn, allow us to write (simpler) partial differential equations models.

In what follows, we first present our models in the case when the state variables for the "small" players lie in a finite set. We next give a few samples of the mathematical results that can be proved for such models. And we finally introduce the models for continuous state variables.

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