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On the boundary behaviour of derivatives of functions in the disc algebra

Sur le comportement au bord de dérivées de fonctions de l'algèbre du disque

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A R T I C L E   I N F O

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A B S T R A C T

We provide with a simple and explicit construction of a function in the disc algebra, whose derivatives enjoy a disjoint universal property near the boundary. The set of functions with such property is topologically generic, densely lineable, and spaceable.

R É S U M É

On présente une construction simple et explicite d'une fonction de l'algèbre du disque dont les dérivés possèdent des propriétés d'universalité disjointe au bord. L'ensemble des fonctions ayant une telle propriété est topologiquement générique et contient un sous-espace dense et un sous-espace fermé de dimension infinie.

1. Introduction and statement of the main result

Let us denote by $\mathbb{T}$ the boundary of the unit disc $\mathbb{D} := \{z \in \mathbb{C},\ |z| < 1\}$ and by $H(\mathbb{D})$ the space of all holomorphic functions in $\mathbb{D}$, endowed with the Fréchet topology of uniform convergence on compacta. The behaviour of functions in $H(\mathbb{D})$ near $\mathbb{T}$ is of crucial interest in complex analysis. It was shown by Bagemihl in 1954 that, for any function φ measurable on $\mathbb{T}$, there exists f in $H(\mathbb{D})$ such that $f(r\zeta) \to \varphi(\zeta)$ as $r \to 1$, for almost every $\zeta \in \mathbb{T}$ [2]. Kahane and Katznelson [11] proved that such functions can have an arbitrary radial growth to the boundary. Later, functions $f \in H(\mathbb{D})$ enjoying the following *universal* property were exhibited [3,8]: given any measurable function φ on $\mathbb{T}$, there exists an increasing sequence $(r_n)_n$, $0 < r_n < 1$, converging to 1, such that, for any $z_0 \in \mathbb{D}$ and almost every $\zeta \in \mathbb{T}$,

$$\lim_{n\to\infty} f(r_n(\zeta - z_0) + z_0) = \varphi(\zeta).$$

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Bayart's result proves that the set of such functions is *generic*, *i.e.* is a countable intersection of open and dense subsets of $H(\mathbb{D})$ [3]. These objects echo other universal objects, such as *universal Taylor series*, which have been much studied during the last two decades: $f = \sum_k a_k z^k \in H(\mathbb{D})$ is a universal Taylor series if given any compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement, and any function g continuous on K and holomorphic in its interior, there exists an increasing sequence $(\lambda_n)_n$ of integers such that $\sum_{k=1}^{\lambda_n} a_k z^k \to g$ uniformly on K as $n \to \infty$. The existence of such functions was proven in [14]. Universal Taylor series were shown to enjoy non-tangential or radial universal properties as above, see [9] and the references therein.

In the known results, building functions with universal boundary behaviour relies on applying complex approximation theorems, like Mergelyan's theorem, and the constructions give functions with wild non-tangential behaviour near large subsets of $\mathbb{T}$. Since these subsets are not explicit, the methods do not permit to prescribe points on $\mathbb{T}$ and to build explicit functions with universal behaviour near these specific points. In this short note, we will provide with a very simple way to build functions in $H(\mathbb{D})$ having universal boundary behaviour near prescribed subsets of $\mathbb{T}$. We will not make use of either Runge's or Mergelyan's theorems, as our construction will be simply based on polynomial interpolation. In particular, it will provide us with explicit functions universal with respect to the prescribed subset of $\mathbb{T}$. Subsequently, the built functions will live in the disc algebra $A(\mathbb{D})$ – the set of analytic functions in $\mathbb{D}$ that are continuous on $\overline{\mathbb{D}}$ – and the universal approximation will be a property of their derivatives.

More precisely, the main result is as follows. We recall that a subset A of a complete metrizable topological vector space X is a dense G_δ-subset of X if it is a countable intersection of dense open sets. It is said to be *densely lineable* if it contains, except 0, a dense subspace in X, and it is called *spaceable* if it contains, except 0, an infinite dimensional closed subspace of X.

Main Theorem. *Let $(\zeta_k)_{k\in\mathbb{N}} \subset \mathbb{T}$ and $(z_n^k)_{n,k\in\mathbb{N}} \subset \mathbb{D}$ be sequences such that $z_n^k \to \zeta_k$ as $n \to \infty$, $1 \le k < \infty$. There exists a function $f \in A(\mathbb{D})$ with the following property: for any sequence $(w_k)_k \subset \mathbb{C}$, there exists an increasing sequence $(n_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ such that, for any $k \ge 1$,*

$$f^{(k)}(z_{n_j}^k) \to w_k \text{ as } j \to \infty.$$

The set of such functions, denoted by $\mathcal{U}((\zeta_k), (z_n^k))$, is a dense G_δ-subset of $A(\mathbb{D})$, and is densely lineable and spaceable.

An important feature is that the sequence $(n_j)_j$ does not depend on k. Observe that the sequences of points converging to each ζ_k are arbitrary, and in particular possibly contained in a curve tangent to $\mathbb{T}$. By the Riemann mapping theorem and its refinement ([12,16]), the theorem easily extends to any bounded simply connected domain with piecewise C^∞ boundary. For such domains, it is an improvement of a result due to Siskaki [17], which asserts that, generically, any function of the disc algebra has unbounded derivatives on $\mathbb{D}$. Obviously, taking $\{\zeta_k, k \ge 1\}$ dense in $\mathbb{T}$, the elements of $\mathcal{U}((\zeta_k), (z_n^k))$ are extendable at no point of $\mathbb{T}$ and totally unbounded. Thus we recover some generic results given in [10,15] for bounded simply connected domain with smooth boundary.

Our main theorem has some operator-theoretic flavour. Indeed, if you denote by $(L_n^k)_n : A(\mathbb{D}) \mapsto \mathbb{C}$ the sequence of continuous linear maps defined by $L_n^k(f) := f^{(k)}(z_n^k)$, $k \ge 1$, this theorem can be reformulated into saying that the family $\{(L_n^k)_n, k \ge 1\}$ is *disjoint universal* in the sense of [5,7]. We may also add that the search for linear structures in sets of strange functions is a classical topic, *e.g.*, [1,6].

2. Preliminaries

The Main Theorem will be obtained as an application of general results from the theory of universality. Let Y be a separable complete metrizable topological vector space (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and Z a metrizable topological vector space (over $\mathbb{K}$), whose topologies are induced by translation-invariant metrics d_Y and ϱ, respectively. Let $L_n : Y \to Z$, $n \in \mathbb{N}$, be continuous linear mappings.

Definition 2.1. We say that $y \in Y$ is universal with respect to $(L_n)_n$ if

$$Z \subset \overline{\{L_n y : n \in \mathbb{N}\}}.$$

We denote by $\mathcal{U}(L_n)$ the set of such universal elements.

Most of the classical universality results can be viewed as applications of the following theorem.

Theorem 2.2. *(1) ([4, Theorems 26 and 27]) We assume that there exists a dense subset Y_0 of Y such that $(L_n y)_n$ converges to an element in Z for any $y \in Y_0$. Then the following are equivalent:*

(i) $\mathcal{U}(L_n) \neq \emptyset$;
(ii) *for any open subset $U \neq \emptyset$ of Y and any open subset $V \neq \emptyset$ of Z, there is some $n \in \mathbb{N}$ with $L_n(U) \cap V \neq \emptyset$;*